FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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FHSST Editors

Jaynie Padayachee ; Joanne Boulle ; Diana Mulcahy ; Annette Nell ; René Toerien ; Donovan Whitfield

FHSST Contributors

Rory Adams ; Prashant Arora ; Richard Baxter ; Dr. Sarah Blyth ; Sebastian Bodenstein ; Graeme Broster ; Richard Case ; Brett Cocks ; Tim Crombie ; Dr. Anne Dabrowski ; Laura Daniels ; Sean Dobbs ; Fernando Durrell ; Dr. Dan Dwyer ; Frans van Eeden ; Giovanni Franzoni ; Ingrid von Glehn ; Tamara von Glehn ; Lindsay Glesener ; Dr. Vanessa Godfrey ; Dr. Johan Gonzalez ; Hemant Gopal ; Umeshree Govender ; Heather Gray ; Lynn Greeff ; Dr. Tom Gutierrez ; Brooke Haag ; Kate Hadley ; Dr. Sam Halliday ; Asheena Hanuman ; Neil Hart ; Nicholas Hatcher ; Dr. Mark Horner ; Mfandaidza Hove ; Robert Hovden ; Jennifer Hsieh ; Clare Johnson ; Luke Jordan ; Tana Joseph ; Dr. Jennifer Klay ; Lara Kruger ; Sihle Kubheka ; Andrew Kubik ; Dr. Marco van Leeuwen ; Dr. Anton Machacek ; Dr. Komal Maheshwari ; Kosma von Maltitz ; Nicole Masureik ; John Mathew ; JoEllen McBride ; Nikolai Meures ; Riana Meyer ; Jenny Miller ; Abdul Mirza ; Asogan Moodaly ; Jothi Moodley ; Nolene Naidu ; Tyrone Negus ; Thomas O'Donnell ; Dr. Markus Oldenburg ; Dr. Jaynie Padayachee ; Nicolette Pekeur ; Sirika Pillay ; Jacques Plaut ; Andrea Prinsloo ; Joseph Raimondo ; Sanya Rajani ; Prof. Sergey Rakityansky ; Alastair Ramlakan ; Razvan Remsing ; Max Richter ; Sean Riddle ; Evan Robinson ; Dr. Andrew Rose ; Bianca Ruddy ; Katie Russell ; Duncan Scott ; Helen Seals ; Ian Sherratt ; Roger Sieloff ; Bradley Smith ; Greg Solomon ; Mike Stringer ; Shen Tian ; Robert Torregrosa ; Jimmy Tseng ; Helen Waugh ; Dr. Dawn Webber ; Michelle Wen ; Dr. Alexander Wetzler ; Dr. Spencer Wheaton ; Vivian White ; Dr. Gerald Wigger ; Harry Wiggins ; Wendy Williams ; Julie Wilson ; Andrew Wood ; Emma Wormauld ; Sahal Yacoob ; Jean Youssef

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Part I

Basics

Chapter 1

Introduction to Book

1.1 The Language of Mathematics

The purpose of any language, like English or Zulu, is to make it possible for people to communicate. All languages have an alphabet, which is a group of letters that are used to make up words. There are also rules of grammar which explain how words are supposed to be used to build up sentences. This is needed because when a sentence is written, the person reading the sentence understands exactly what the writer is trying to explain. Punctuation marks (like a full stop or a comma) are used to further clarify what is written.

Mathematics is a language, specifically it is the language of Science. Like any language, mathematics has letters (known as numbers) that are used to make up words (known as expressions), and sentences (known as equations). The punctuation marks of mathematics are the different signs and symbols that are used, for example, the plus sign ($+$), the minus sign (-), the multiplication sign ($\times$), the equals sign ($=$) and so on. There are also rules that explain how the numbers should be used together with the signs to make up equations that express some meaning.

Part II

Grade 10

Chapter 2

Review of Past Work

2.1 Introduction

This chapter describes some basic concepts which you have seen in earlier grades, and lays the foundation for the remainder of this book. You should feel confident with the content in this chapter, before moving on with the rest of the book.

So try out your skills on the exercises throughout this chapter and ask your teacher for more questions just like them. You can also try making up your own questions, solve them and try them out on your classmates to see if you get the same answers.

Practice is the only way to get good at maths!

2.2 What is a number?

A number is a way to represent quantity. Numbers are not something that you can touch or hold, because they are not physical. But you can touch three apples, three pencils, three books. You can never just touch three, you can only touch three of something. However, you do not need to see three apples in front of you to know that if you take one apple away, that there will be two apples left. You can just think about it. That is your brain representing the apples in numbers and then performing arithmetic on them.

A number represents quantity because we can look at the world around us and quantify it using numbers. How many minutes? How many kilometers? How many apples? How much money? How much medicine? These are all questions which can only be answered using numbers to tell us "how much" of something we want to measure.

A number can be written many different ways and it is always best to choose the most appropriate way of writing the number. For example, "a half" may be spoken aloud or written in words, but that makes mathematics very difficult and also means that only people who speak the same language as you can understand what you mean. A better way of writing "a half" is as a fraction $\frac{1}{2}$ or as a decimal number $0{,}5$. It is still the same number, no matter which way you write it.

In high school, all the numbers which you will see are called *real numbers* and mathematicians use the symbol $\mathbb{R}$ to stand for the *set of all real numbers*, which simply means all of the real numbers. Some of these real numbers can be written in a particular way and some cannot. Different types of numbers are described in detail in Section 1.12.

2.3 Sets

A *set* is a group of objects with a well-defined criterion for membership. For example, the criterion for belonging to a set of apples, is that it must be an apple. The set of apples can then be divided into red apples and green apples, but they are all still apples. All the red apples form another set which is a *sub-set* of the set of apples. A sub-set is part of a set. All the green apples form another sub-set.

Now we come to the idea of a **union**, which is used to combine things. The symbol for **union** is $\cup$. Here we use it to combine two or more intervals. For example, if x is a real number such that $1 < x \leq 3$ or $6 \leq x < 10$, then the set of all the possible x values is

$$(1{,}3] \cup [6{,}10) \quad (2.1)$$

where the $\cup$ sign means the union (or combination) of the two intervals. We use the set and interval notation and the symbols described because it is easier than having to write everything out in words.

2.4 Letters and Arithmetic

The simplest things that can be done with numbers is to add, subtract, multiply or divide them. When two numbers are added, subtracted, multiplied or divided, you are performing *arithmetic*[1]. These four basic operations can be performed on any two real numbers.

Mathematics as a language uses special notation to write things down. So instead of:

one plus one is equal to two

mathematicians write

$$1 + 1 = 2$$

In earlier grades, place holders were used to indicate missing numbers in an equation.

$$1 + \square = 2$$
$$4 - \square = 2$$
$$\square + 3 - 2\square = 2$$

However, place holders only work well for simple equations. For more advanced mathematical workings, letters are usually used to represent numbers.

$$1 + x = 2$$
$$4 - y = 2$$
$$z + 3 - 2z = 2$$

These letters are referred to as **variables**, since they can take on any value depending on what is required. For example, $x = 1$ in Equation 2.2, but $x = 26$ in $2 + x = 28$.
A **constant** has a fixed value. The number 1 is a constant. The *speed of light* in a vacuum is also a constant which has been defined to be exactly 299 792 458 m·s^{-1}(read metres per second). The speed of light is a big number and it takes up space to always write down the entire number. Therefore, letters are also used to represent some constants. In the case of the speed of light, it is accepted that the letter c represents the speed of light. Such constants represented by letters occur most often in physics and chemistry.

Additionally, letters can be used to describe a situation, mathematically. For example, the following equation

$$x + y = z \quad (2.2)$$

can be used to describe the situation of finding how much change can be expected for buying an item. In this equation, y represents the price of the item you are buying, x represents the amount of change you should get back and z is the amount of money given to the cashier. So, if the price is R10 and you gave the cashier R15, then write R15 instead of z and R10 instead of y and the change is then x.

$$x + 10 = 15 \quad (2.3)$$

We will learn how to "solve" this equation towards the end of this chapter.

[1] Arithmetic is derived from the Greek word *arithmos* meaning *number*.

2.5 Addition and Subtraction

Addition (+) and subtraction (-) are the most basic operations between numbers but they are very closely related to each other. You can think of subtracting as being the opposite of adding since adding a number and then subtracting the same number will not change what you started with. For example, if we start with a and add b, then subtract b, we will just get back to a again

$$\begin{aligned} a+b-b &= a \\ 5+2-2 &= 5 \end{aligned} \tag{2.4}$$

If we look at a number line, then addition means that we move to the right and subtraction means that we move to the left.

The order in which numbers are added does not matter, but the order in which numbers are subtracted does matter. This means that:

$$\begin{aligned} a+b &= b+a \\ a-b &\neq b-a \quad \text{if } a \neq b \end{aligned} \tag{2.5}$$

The sign $\neq$ means "is not equal to". For example, $2+3=5$ and $3+2=5$, but $5-3=2$ and $3-5=-2$. -2 is a negative number, which is explained in detail in Section 2.8.

Extension: Commutativity for Addition
The fact that $a+b=b+a$, is known as the *commutative* property for addition.

2.6 Multiplication and Division

Just like addition and subtraction, multiplication ($\times$, $\cdot$) and division ($\div$, /) are opposites of each other. Multiplying by a number and then dividing by the same number gets us back to the start again:

$$\begin{aligned} a \times b \div b &= a \\ 5 \times 4 \div 4 &= 5 \end{aligned} \tag{2.6}$$

Sometimes you will see a multiplication of letters as a dot or without any symbol. Don't worry, its exactly the same thing. Mathematicians are lazy and like to write things in the shortest, neatest way possible.

$$\begin{aligned} abc &= a \times b \times c \\ a \cdot b \cdot c &= a \times b \times c \end{aligned} \tag{2.7}$$

It is usually neater to write known numbers to the left, and letters to the right. So although $4x$ and $x4$ are the same thing, it looks better to write $4x$. In this case, the "4" is a constant that is referred to as the *coefficient* of x.

Extension: Commutativity for Multiplication
The fact that $ab=ba$ is known as the *commutative* property of multiplication. Therefore, both addition and multiplication are described as commutative operations.

2.7 Brackets

Brackets[2] in mathematics are used to show the order in which you must do things. This is important as you can get different answers depending on the order in which you do things. For

[2]Sometimes people say "parenthesis" instead of "brackets".

example

$$(5 \times 5) + 20 = 45 \tag{2.8}$$

whereas

$$5 \times (5 + 20) = 125 \tag{2.9}$$

If there are no brackets, you should always do multiplications and divisions first and then additions and subtractions[3]. You can always put your own brackets into equations using this rule to make things easier for yourself, for example:

$$\begin{aligned} a \times b + c \div d &= (a \times b) + (c \div d) \\ 5 \times 5 + 20 \div 4 &= (5 \times 5) + (20 \div 4) \end{aligned} \tag{2.10}$$

If you see a multiplication outside a bracket like this

$$\begin{aligned} &a(b + c) \\ &3(4 - 3) \end{aligned} \tag{2.11}$$

then it means you have to multiply each part inside the bracket by the number outside

$$\begin{aligned} a(b + c) &= ab + ac \\ 3(4 - 3) &= 3 \times 4 - 3 \times 3 = 12 - 9 = 3 \end{aligned} \tag{2.12}$$

unless you can simplify everything inside the bracket into a single term. In fact, in the above example, it would have been smarter to have done this

$$3(4 - 3) = 3 \times (1) = 3 \tag{2.13}$$

It can happen with letters too

$$3(4a - 3a) = 3 \times (a) = 3a \tag{2.14}$$

Extension: Distributivity

The fact that $a(b + c) = ab + ac$ is known as the *distributive* property.

If there are two brackets multiplied by each other, then you can do it one step at a time

$$\begin{aligned} (a + b)(c + d) &= a(c + d) + b(c + d) \\ &= ac + ad + bc + bd \\ (a + 3)(4 + d) &= a(4 + d) + 3(4 + d) \\ &= 4a + ad + 12 + 3d \end{aligned} \tag{2.15}$$

2.8 Negative Numbers

2.8.1 What is a negative number?

Negative numbers can be very confusing to begin with, but there is nothing to be afraid of. The numbers that are used most often are greater than zero. These numbers are known as *positive numbers*.

A negative number is simply a number that is less than zero. So, if we were to take a positive number a and subtract it from zero, the answer would be the negative of a.

$$0 - a = -a$$

[3]Multiplying and dividing can be performed in any order as it doesn't matter. Likewise it doesn't matter which order you do addition and subtraction. Just as long as you do any $\times \div$ before any $+-$.

On a number line, a negative number appears to the left of zero and a positive number appears to the right of zero.

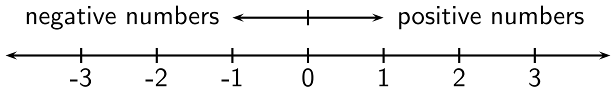


Figure 2.1: On the number line, numbers increase towards the right and decrease towards the left. Positive numbers appear to the right of zero and negative numbers appear to the left of zero.

2.8.2 Working with Negative Numbers

When you are adding a negative number, it is the same as subtracting that number if it were positive. Likewise, if you subtract a negative number, it is the same as adding the number if it were positive. Numbers are either positive or negative, and we call this their *sign*. A positive number has positive sign (+), and a negative number has a negative sign (-).

Subtraction is actually the same as adding a *negative number*.

In this example, a and b are positive numbers, but $-b$ is a negative number

$$\begin{aligned} a - b &= a + (-b) \\ 5 - 3 &= 5 + (-3) \end{aligned} \tag{2.16}$$

So, this means that subtraction is simply a short-cut for adding a negative number, and instead of writing $a + (-b)$, we write $a - b$. This also means that $-b + a$ is the same as $a - b$. Now, which do you find easier to work out?

Most people find that the first way is a bit more difficult to work out than the second way. For example, most people find $12 - 3$ a lot easier to work out than $-3 + 12$, even though they are the same thing. So, $a - b$, which looks neater and requires less writing, is the accepted way of writing subtractions.

Table 2.1 shows how to calculate the sign of the answer when you multiply two numbers together. The first column shows the sign of the first number, the second column gives the sign of the second number, and the third column shows what sign the answer will be. So multiplying or

a	b	$a \times b$ or $a \div b$
+	+	+
+	-	-
-	+	-
-	-	+

Table 2.1: Table of signs for multiplying or dividing two numbers.

dividing a negative number by a positive number always gives you a negative number, whereas multiplying or dividing numbers which have the same sign always gives a positive number. For example, $2 \times 3 = 6$ and $-2 \times -3 = 6$, but $-2 \times 3 = -6$ and $2 \times -3 = -6$.

Adding numbers works slightly differently, have a look at Table 2.2. The first column shows the sign of the first number, the second column gives the sign of the second number, and the third column shows what sign the answer will be.

a	b	$a + b$
+	+	+
+	-	?
-	+	?
-	-	-

Table 2.2: Table of signs for adding two numbers.

If you add two positive numbers you will always get a positive number, but if you add two negative numbers you will always get a negative number. If the numbers have different sign, then the sign of the answer depends on which one is bigger.

2.8.3 Living Without the Number Line

The number line in Figure 2.1 is a good way to visualise what negative numbers are, but it can get very inefficient to use it every time you want to add or subtract negative numbers. To keep things simple, we will write down three tips that you can use to make working with negative numbers a little bit easier. These tips will let you work out what the answer is when you add or subtract numbers which may be negative and will also help you keep your work tidy and easier to understand.

Negative Numbers Tip 1

If you are given an equation like $-a+b$, then it is easier to move the numbers around so that the equation looks easier. For this case, we have seen that adding a negative number to a positive number is the same as subtracting the number from the positive number. So,

$$\begin{aligned} -a+b &= b-a \\ -5+10 &= 10-5=5 \end{aligned} \tag{2.17}$$

This makes equations easier to understand. For example, a question like "What is $-7+11$?" looks a lot more complicated than "What is $11-7$?", even though they are exactly the same question.

Negative Numbers Tip 2

When you have two negative numbers like $-3-7$, you can calculate the answer by simply adding together the numbers as if they were positive and then putting a negative sign in front.

$$\begin{aligned} -c-d &= -(c+d) \\ -7-2 &= -(7+2)=-9 \end{aligned} \tag{2.18}$$

Negative Numbers Tip 3

In Table 2.2 we saw that the sign of two numbers added together depends on which one is bigger. This tip tells us that all we need to do is take the smaller number away from the larger one, and remember to put a negative sign before the answer if the bigger number was subtracted to begin with. In this equation, F is bigger than e.

$$\begin{aligned} e-F &= -(F-e) \\ 2-11 &= -(11-2)=-9 \end{aligned} \tag{2.19}$$

You can even combine these tips together, so for example you can use Tip 1 on $-10+3$ to get $3-10$, and then use Tip 3 to get $-(10-3)=-7$.

Exercise: Negative Numbers

1. Calculate:
   (a) $(-5)-(-3)$ (b) $(-4)+2$ (c) $(-10)\div(-2)$
   (d) $11-(-9)$ (e) $-16-(6)$ (f) $-9\div 3\times 2$
   (g) $(-1)\times 24\div 8\times(-3)$ (h) $(-2)+(-7)$ (i) $1-12$
   (j) $3-64+1$ (k) $-5-5-5$ (l) $-6+25$
   (m) $-9+8-7+6-5+4-3+2-1$

2. Say whether the sign of the answer is + or -

   (a) $-5+6$ (b) $-5+1$ (c) $-5 \div -5$
   (d) $-5 \div 5$ (e) $5 \div -5$ (f) $5 \div 5$
   (g) -5×-5 (h) -5×5 (i) 5×-5
   (j) 5×5

2.9 Rearranging Equations

Now that we have described the basic rules of negative and positive numbers and what to do when you add, subtract, multiply and divide them, we are ready to tackle some real mathematics problems!

Earlier in this chapter, we wrote a general equation for calculating how much change (x) we can expect if we know how much an item costs (y) and how much we have given the cashier (z). The equation is:

$$x + y = z \tag{2.20}$$

So, if the price is R10 and you gave the cashier R15, then write R15 instead of z and R10 instead of y.

$$x + 10 = 15 \tag{2.21}$$

Now, that we have written this equation down, how exactly do we go about finding what the change is? In mathematical terms, this is known as solving an equation for an unknown (x in this case). We want to re-arrange the terms in the equation, so that only x is on the left hand side of the = sign and everything else is on the right.

The most important thing to remember is that an equation is like a set of weighing scales. In order to keep the scales balanced, whatever, is done to one side, must be done to the other.

Method: Rearranging Equations

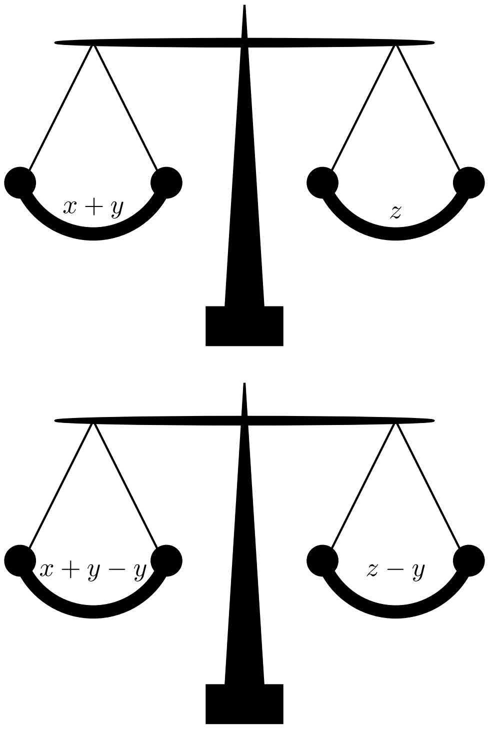
You can add, subtract, multiply or divide both sides of an equation by any number you want, as long as you always do it to both sides.

So for our example we could subtract y from both sides

$$\begin{aligned} x + y &= z \\ x + y - y &= z - y \\ x &= z - y \\ x &= 15 - 10 \\ &= 5 \end{aligned} \tag{2.22}$$

so now we can find the change is the price subtracted from the amount handed over to the cashier. In the example, the change should be R5. In real life we can do this in our head, the human brain is very smart and can do arithmetic without even knowing it.

When you subtract a number from both sides of an equation, it looks just like you moved a positive number from one side and it became a negative on the other, which is exactly what happened. Likewise if you move a multiplied number from one side to the other, it looks like it changed to a divide. This is because you really just divided both sides by that number, and a



divide the other side too.

Figure 2.2: An equation is like a set of weighing scales. In order to keep the scales balanced, you must do the same thing to both sides. So, if you add, subtract, multiply or divide the one side, you must add, subtract, multiply or
divide the other side too.

number divided by itself is just 1

$$\begin{aligned} a(5+c) &= 3a \\ a(5+c) \div a &= 3a \div a \\ \frac{a}{a} \times (5+c) &= 3 \times \frac{a}{a} \\ 1 \times (5+c) &= 3 \times 1 \\ 5+c &= 3 \\ c &= 3-5=-2 \end{aligned} \tag{2.23}$$

However you must be careful when doing this, as it is easy to make mistakes.

The following is the wrong thing to do

$$\begin{aligned} 5a+c &= 3a \\ 5+c &\neq^{4} 3a \div a \end{aligned} \tag{2.24}$$

Can you see why it is wrong? It is wrong because we did not divide the c term by a as well. The correct thing to do is

$$\begin{aligned} 5a+c &= 3a \\ 5+c \div a &= 3 \\ c \div a &= 3-5=-2 \end{aligned} \tag{2.25}$$

Exercise: Rearranging Equations

1. If $3(2r - 5) = 27$, then $2r - 5 =$ .....
2. Find the value for x if $0{,}5(x - 8) = 0{,}2x + 11$
3. Solve $9 - 2n = 3(n + 2)$
4. Change the formula $P = A + Akt$ to $A =$
5. Solve for x: $\frac{1}{ax} + \frac{1}{bx} = 1$

2.10 Fractions and Decimal Numbers

A fraction is one number divided by another number. There are several ways to write a number divided by another one, such as $a \div b$, a/b and $\frac{a}{b}$. The first way of writing a fraction is very hard to work with, so we will use only the other two. We call the number on the top, the *numerator* and the number on the bottom the *denominator*. For example,

$$\frac{1}{5} \qquad \frac{\text{numerator} = 1}{\text{denominator} = 5} \tag{2.26}$$

Extension: Definition - Fraction
The word *fraction* means *part of a whole*.

The *reciprocal* of a fraction is the fraction turned upside down, in other words the numerator becomes the denominator and the denominator becomes the numerator. So, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.
A fraction multiplied by its reciprocal is always equal to 1 and can be written

$$\frac{a}{b} \times \frac{b}{a} = 1 \tag{2.27}$$

This is because dividing by a number is the same as multiplying by its reciprocal.

Extension: Definition - Multiplicative Inverse
The reciprocal of a number is also known as the multiplicative inverse.

A decimal number is a number which has an integer part and a fractional part. The integer and the fractional parts are separated by a *decimal point*, which is written as a comma in South Africa. For example the number $3\frac{14}{100}$ can be written much more cleanly as $3{,}14$.

All real numbers can be written as a decimal number. However, some numbers would take a huge amount of paper (and ink) to write out in full! Some decimal numbers will have a number which will repeat itself, such as $0{,}33333\ldots$ where there are an infinite number of 3's. We can write this decimal value by using a dot above the repeating number, so $0{,}\dot{3} = 0{,}33333\ldots$. If there are two repeating numbers such as $0{,}121212\ldots$ then you can place dots[5] on each of the repeated numbers $0{,}\dot{1}\dot{2} = 0{,}121212\ldots$. These kinds of repeating decimals are called *recurring decimals*.

Table 2.3 lists some common fractions and their decimal forms.

[5] or a bar, like $0{,}\overline{12}$

Fraction	Decimal Form
$\frac{1}{20}$	0,05
$\frac{1}{16}$	0,0625
$\frac{1}{10}$	0,1
$\frac{1}{8}$	0,125
$\frac{1}{6}$	$0{,}16\dot{6}$
$\frac{1}{5}$	0,2
$\frac{1}{2}$	0,5
$\frac{3}{4}$	0,75

Table 2.3: Some common fractions and their equivalent decimal forms.

2.11 Scientific Notation

In science one often needs to work with very large or very small numbers. These can be written more easily in scientific notation, which has the general form

$$a \times 10^{m} \tag{2.28}$$

where a is a decimal number between 0 and 10 that is rounded off to a few decimal places. The m is an integer and if it is positive it represents how many zeros should appear to the right of a. If m is negative then it represents how many times the decimal place in a should be moved to the left. For example $3{,}2 \times 10^{3}$ represents 32000 and $3{,}2 \times 10^{-3}$ represents $0{,}0032$.

If a number must be converted into scientific notation, we need to work out how many times the number must be multiplied or divided by 10 to make it into a number between 1 and 10 (i.e. we need to work out the value of the exponent m) and what this number is (the value of a). We do this by counting the number of decimal places the decimal point must move.

For example, write the speed of light which is 299 792 458 ms^{-1} in scientific notation, to two decimal places. First, determine where the decimal point must go for two decimal places (to find a) and then count how many places there are after the decimal point to determine m.

In this example, the decimal point must go after the first 2, but since the number after the 9 is a 7, $a = 3{,}00$.

So the number is $3{,}00 \times 10^{m}$, where $m = 8$, because there are 8 digits left after the decimal point. So the speed of light in scientific notation, to two decimal places is $3{,}00 \times 10^{8} ms^{-1}$.

As another example, the size of the HI virus is around 120×10^{-9} m. This is equal to $120 \times 0{,}000000001$ m which is 0,00000012 m.

2.12 Real Numbers

Now that we have learnt about the basics of mathematics, we can look at what real numbers are in a little more detail. The following are examples of real numbers and it is seen that each number is written in a different way.

$$\sqrt{3}, \quad 1{,}2557878, \quad \frac{56}{34}, \quad 10, \quad 2{,}1, \quad -5, \quad -6{,}35, \quad -\frac{1}{90} \tag{2.29}$$

Depending on how the real number is written, it can be further labelled as either rational, irrational, integer or natural. A set diagram of the different number types is shown in Figure 2.3.

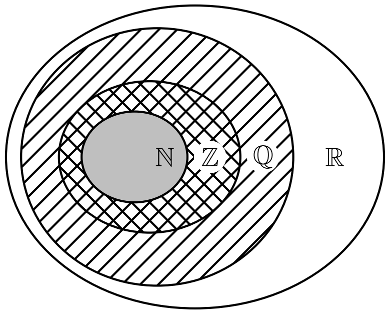


Figure 2.3: Set diagram of all the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$, the integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$. The irrational numbers are the numbers not inside the set of rational numbers. All of the integers are also rational numbers, but not all rational numbers are integers.

Extension: Non-Real Numbers
All numbers that are not real numbers have *imaginary* components. We will not see imaginary numbers in this book but you will see that they come from $\sqrt{-1}$. Since we won't be looking at numbers which are not real, if you see a number you can be sure it is a real one.

2.12.1 Natural Numbers

The first type of numbers that are learnt about are the numbers that were used for counting. These numbers are called *natural numbers* and are the simplest numbers in mathematics.

$$0, 1, 2, 3, 4 \ldots \tag{2.30}$$

Mathematicians use the symbol $\mathbb{N}$ to mean the *set of all natural numbers*. The natural numbers are a *subset* of the real numbers since every natural number is also a real number.

2.12.2 Integers

The integers are all of the natural numbers and their negatives

$$\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \tag{2.31}$$

Mathematicians use the symbol $\mathbb{Z}$ to mean *the set of all integers*. The integers are a subset of the real numbers, since every integer is a real number.

2.12.3 Rational Numbers

The natural numbers and the integers are only able to describe quantities that are whole or complete. For example you can have 4 apples, but what happens when you divide one apple into 4 equal pieces and share it among your friends? Then it is not a whole apple anymore and a different type of number is needed to describe the apples. This type of number is known as a rational number.

A rational number is any number which can be written as:

$$\frac{a}{b} \tag{2.32}$$

where a and b are integers and $b \neq 0$.

The following are examples of rational numbers:

$$\frac{20}{9}, \quad \frac{-1}{2}, \quad \frac{20}{10}, \quad \frac{3}{15} \tag{2.33}$$

Extension: Notation Tip
Rational numbers are any number that can be expressed in the form $\frac{a}{b}; a, b \in \mathbb{Z}; b \neq 0$ which means "the set of numbers $\frac{a}{b}$ when a and b are integers".

Mathematicians use the symbol $\mathbb{Q}$ to mean *the set of all rational numbers*. The set of rational numbers contains all numbers which can be written as terminating or repeating decimals.

Extension: Rational Numbers
All integers are rational numbers with denominator 1.

You can add and multiply rational numbers and still get a rational number at the end, which is very useful. If we have 4 integers, a, b, c and d, then the rules for adding and multiplying rational numbers are

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad (2.34)$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad (2.35)$$

Extension: Notation Tip
The statement "4 integers a, b, c and d" can be written formally as $\{a, b, c, d\} \in \mathbb{Z}$ because the $\in$ symbol means *in* and we say that a, b, c and d are *in* the set of integers.

Two rational numbers ($\frac{a}{b}$ and $\frac{c}{d}$) represent the same number if $ad = bc$. It is always best to simplify any rational number so that the denominator is as small as possible. This can be achieved by dividing both the numerator and the denominator by the same integer. For example, the rational number $1000/10000$ can be divided by 1000 on the top and the bottom, which gives $1/10$. $\frac{2}{3}$ of a pizza is the same as $\frac{8}{12}$ (Figure 2.4).

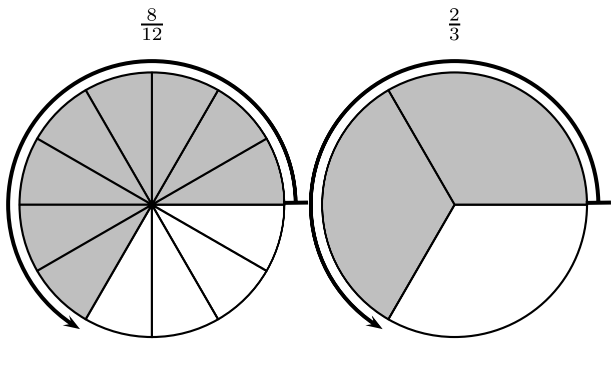


Figure 2.4: $\frac{8}{12}$ of the pizza is the same as $\frac{2}{3}$ of the pizza.

You can also add rational numbers together by finding a *lowest common denominator* and then adding the numerators. Finding a lowest common denominator means finding the lowest number that both denominators are a *factor*[6] of. A factor of a number is an integer which evenly divides that number without leaving a remainder. The following numbers all have a factor of 3

$$3, 6, 9, 12, 15, 18, 21, 24 \ldots$$

and the following all have factors of 4

$$4, 8, 12, 16, 20, 24, 28 \ldots$$

[6] Some people say *divisor* instead of factor.

The common denominators between 3 and 4 are all the numbers that appear in both of these lists, like 12 and 24. The lowest common denominator of 3 and 4 is the number that has both 3 and 4 as factors, which is 12.

For example, if we wish to add $\frac{3}{4}+\frac{2}{3}$, we first need to write both fractions so that their denominators are the same by finding the lowest common denominator, which we know is 12. We can do this by multiplying $\frac{3}{4}$ by $\frac{3}{3}$ and $\frac{2}{3}$ by $\frac{4}{4}$. $\frac{3}{3}$ and $\frac{4}{4}$ are really just complicated ways of writing 1. Multiplying a number by 1 doesn't change the number.

$$\begin{aligned}\frac{3}{4}+\frac{2}{3} &= \frac{3}{4}\times\frac{3}{3}+\frac{2}{3}\times\frac{4}{4} \\ &= \frac{3\times 3}{4\times 3}+\frac{2\times 4}{3\times 4} \\ &= \frac{9}{12}+\frac{8}{12} \\ &= \frac{9+8}{12} \\ &= \frac{17}{12}\end{aligned} \tag{2.36}$$

Dividing by a rational number is the same as multiplying by its reciprocal, as long as neither the numerator nor the denominator is zero:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}.\frac{d}{c}=\frac{ad}{bc} \tag{2.37}$$

A rational number may be a *proper* or *improper* fraction.

Proper fractions have a numerator that is smaller than the denominator. For example,

$$\frac{-1}{2},\frac{3}{15},\frac{-5}{-20}$$

are proper fractions.

Improper fractions have a numerator that is larger than the denominator. For example,

$$\frac{-10}{2},\frac{13}{15},\frac{-53}{-20}$$

are improper fractions. Improper fractions can always be written as the sum of an integer and a proper fraction.

Converting Rationals into Decimal Numbers

Converting rationals into decimal numbers is very easy.

If you use a calculator, you can simply divide the numerator by the denominator.

If you do not have a calculator, then you unfortunately have to use long division.

Since long division, was first taught in primary school, it will not be discussed here. If you have trouble with long division, then please ask your friends or your teacher to explain it to you.

2.12.4 Irrational Numbers

An *irrational number* is any real number that is not a rational number. When expressed as decimals these numbers can never be fully written out as they have an infinite number of decimal places which never fall into a repeating pattern, for example $\sqrt{2}=1{,}41421356\ldots$, $\pi=3{,}14159265\ldots$. π is a Greek letter and is pronounced "pie".

Exercise: Real Numbers

1. Identify the number type (rational, irrational, real, integer) of each of the following numbers:
   (a) $\frac{c}{d}$ if c is an integer and if d is irrational.
   (b) $\frac{3}{2}$
   (c) -25
   (d) 1,525
   (e) $\sqrt{10}$
2. Is the following pair of numbers real and rational or real and irrational? Explain.
   $\sqrt{4}; \frac{1}{8}$

2.13 Mathematical Symbols

The following is a table of the meanings of some mathematical signs and symbols that you should have come across in earlier grades.

Sign or Symbol	Meaning
$>$	greater than
$<$	less than
$\geq$	greater than or equal to
$\leq$	less than or equal to

So if we write $x > 5$, we say that x is greater than 5 and if we write $x \geq y$, we mean that x can be greater than or equal to y. Similarly, $<$ means 'is less than' and $\leq$ means 'is less than or equal to'. Instead of saying that x is between 6 and 10, we often write $6 < 10$. This directly means 'six is less than x which in turn is less than ten'.

Exercise: Mathematical Symbols

1. Write the following in symbols:
   (a) x is greater than 1
   (b) y is less than or equal to z
   (c) a is greater than or equal to 21
   (d) p is greater than or equal to 21 and p is less than or equal to 25

2.14 Infinity

Infinity (symbol ∞) is usually thought of as something like "the largest possible number" or "the furthest possible distance". In mathematics, infinity is often treated as if it were a number, but it is clearly a very different type of "number" than the integers or reals.

When talking about recurring decimals and irrational numbers, the term *infinite* was used to describe *never-ending* digits.

2.15 End of Chapter Exercises

1. Calculate
   (a) $18 - 6 \times 2$
   (b) $10 + 3(2 + 6)$
   (c) $50 - 10(4 - 2) + 6$
   (d) $2 \times 9 - 3(6 - 1) + 1$
   (e) $8 + 24 \div 4 \times 2$
   (f) $30 - 3 \times 4 + 2$
   (g) $36 \div 4(5 - 2) + 6$
   (h) $20 - 4 \times 2 + 3$
   (i) $4 + 6(8 + 2) - 3$
   (j) $100 - 10(2 + 3) + 4$
2. If $p = q + 4r$, then $r =$ .....
3. Solve $\frac{x-2}{3} = x - 3$

Chapter 3

Rational Numbers - Grade 10

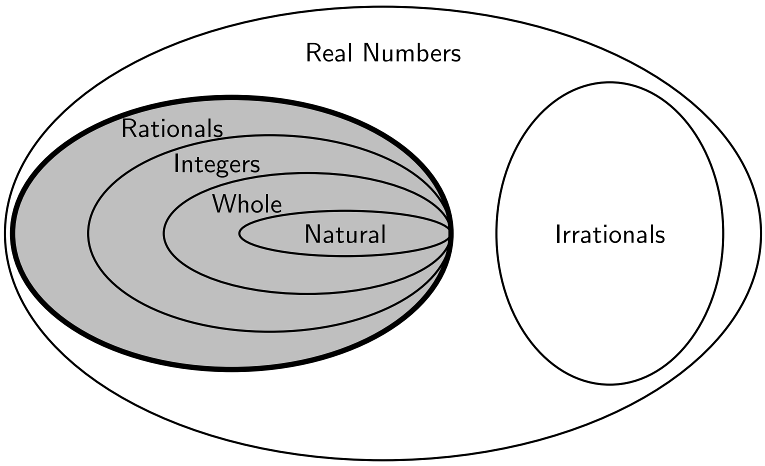
3.1 Introduction

As described in Chapter 2, a number is a way of representing quantity. The numbers that will be used in high school are all real numbers, but there are many different ways of writing any single real number.

This chapter describes *rational numbers*.

3.2 The Big Picture of Numbers

All numbers inside the grey oval are rational numbers.



3.3 Definition

The following numbers are all rational numbers.

$$\frac{10}{1}, \quad \frac{21}{7}, \quad \frac{-1}{-3}, \quad \frac{10}{20}, \quad \frac{-3}{6} \tag{3.1}$$

You can see that all the denominators and all the numerators are integers[1].

Definition: Rational Number
A rational number is any number which can be written as:

$$\frac{a}{b} \tag{3.2}$$

where a and b are integers and $b \neq 0$.

[1] Integers are the counting numbers (1, 2, 3, ...), their opposites (-1, -2, -3, ...), and 0.

Important: Only fractions which have a numerator and a denominator that are integers are rational numbers.

This means that all integers are rational numbers, because they can be written with a denominator of 1.

Therefore, while

$$\frac{\sqrt{2}}{7}, \quad \frac{-1{,}33}{-3}, \quad \frac{\pi}{20}, \quad \frac{-3}{6{,}39} \tag{3.3}$$

are **not examples** of rational numbers, because in each case, either the numerator or the denominator is not an integer.

Exercise: Rational Numbers

1. If a is an integer, b is an integer and c is not an integer, which of the following are rational numbers:

   (a) $\frac{5}{6}$ (b) $\frac{a}{3}$ (c) $\frac{b}{2}$ (d) $\frac{1}{c}$

2. If $\frac{a}{1}$ is a rational number, which of the following are valid values for a?

   (a) 1 (b) -10 (c) $\sqrt{2}$ (d) $2{,}1$

3.4 Forms of Rational Numbers

All integers and fractions with integer numerators and denominators are rational numbers. There are two more forms of rational numbers.

Activity :: Investigation : Decimal Numbers

You can write the rational number $\frac{1}{2}$ as the decimal number 0,5. Write the following numbers as decimals:

1. $\frac{1}{4}$
2. $\frac{1}{10}$
3. $\frac{2}{5}$
4. $\frac{1}{100}$
5. $\frac{2}{3}$

Do the numbers after the decimal comma end or do they continue? If they continue, is there a repeating pattern to the numbers?

You can write a rational number as a decimal number. Therefore, you should be able to write a decimal number as a rational number. Two types of decimal numbers can be written as rational numbers:

1. decimal numbers that end or *terminate*, for example the fraction $\frac{4}{10}$ can be written as 0,4.

2. decimal numbers that have a repeating pattern of numbers, for example the fraction $\frac{1}{3}$ can be written as $0{,}33333\overline{3}$.

For example, the rational number $\frac{5}{6}$ can be written in decimal notation as $0{,}8333\overline{3}$, and similarly, the decimal number 0,25 can be written as a rational number as $\frac{1}{4}$.

Important: Notation for Repeating Decimals

You can use a bar over the repeated numbers to indicate that the decimal is a repeating decimal.

3.5 Converting Terminating Decimals into Rational Numbers

A decimal number has an integer part and a fractional part. For example, $10{,}589$ has an integer part of 10 and a fractional part of $0{,}589$ because $10 + 0{,}589 = 10{,}589$. The fractional part can be written as a rational number, i.e. with a numerator and a denominator that are integers.

Each digit after the decimal point is a fraction with denominator in increasing powers of ten. For example:

- $\frac{1}{10}$ is $0{,}1$
- $\frac{1}{100}$ is $0{,}01$

This means that:

$$\begin{aligned} 2{,}103 &= 2 + \frac{1}{10} + \frac{0}{100} + \frac{3}{1000} \\ &= 2\frac{103}{1000} \\ &= \frac{2103}{1000} \end{aligned}$$

Exercise: Fractions

1. Write the following as fractions:

   (a) $0{,}1$ (b) $0{,}12$ (c) $0{,}58$ (d) $0{,}2589$

3.6 Converting Repeating Decimals into Rational Numbers

When the decimal is a repeating decimal, a bit more work is needed to write the fractional part of the decimal number as a fraction. We will explain by means of an example.

If we wish to write $0{,}\overline{3}$ in the form $\frac{a}{b}$ (where a and b are integers) then we would proceed as follows

$$\begin{aligned} x &= 0{,}33333\ldots && (3.4) \\ 10x &= 3{,}33333\ldots \quad \text{multiply by 10 on both sides} && (3.5) \\ 9x &= 3 \quad \text{subtracting (3.4) from (3.5)} \\ x &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

And another example would be to write $5{,}\overline{432}$ as a rational fraction

$$\begin{aligned} x &= 5{,}432432432\ldots && (3.6) \\ 1000x &= 5432{,}432432432\ldots \quad \text{multiply by 1000 on both sides} && (3.7) \\ \\ 999x &= 5427 \quad \text{subtracting (3.6) from (3.7)} \\ x &= \frac{5427}{999} = \frac{201}{37} \end{aligned}$$

For the first example, the decimal number was multiplied by 10 and for the second example, the decimal number was multiplied by 1000. This is because for the first example there was only one number (i.e. 3) that recurred, while for the second example there were three numbers (i.e. 432) that recurred.

In general, if you have one number recurring, then multiply by 10, if you have two numbers recurring, then multiply by 100, if you have three numbers recurring, then multiply by 1000. Can you spot the pattern yet?

The number of zeros after the 1 is the same as the number of recurring numbers.

But not all decimal numbers can be written as rational numbers, because some decimal numbers like $\sqrt{2} = 1{,}4142135...$ is an irrational number and cannot be written with an integer numerator and an integer denominator. However, when possible, you should always use rational numbers or fractions instead of decimals.

Exercise: Repeated Decimal Notation

1. Write the following using the repeated decimal notation:
   (a) $0{,}11111111\ldots$
   (b) $0{,}1212121212\ldots$
   (c) $0{,}123123123123\ldots$
   (d) $0{,}11414541454145\ldots$
2. Write the following in decimal form, using the repeated decimal notation:
   (a) $\frac{2}{3}$
   (b) $1\frac{3}{11}$
   (c) $4\frac{5}{6}$
   (d) $2\frac{1}{9}\ldots$
3. Write the following decimals in fractional form:
   (a) $0{,}6333\ldots$
   (b) $5{,}3131\overline{31}$
   (c) $11{,}570571\ldots$
   (d) $0{,}999999\ldots$

3.7 Summary

The following are rational numbers:

- Fractions with both denominator and numerator as integers.
- Integers.
- Decimal numbers that end.
- Decimal numbers that repeat.

3.8 End of Chapter Exercises

1. If a is an integer, b is an integer and c is not an integer, which of the following are rational numbers:
   (a) $\frac{5}{6}$
   (b) $\frac{a}{3}$
   (c) $\frac{b}{2}$
   (d) $\frac{1}{c}$
2. Write each decimal as a simple fraction:
   (a) $0{,}5$
   (b) $0{,}12$
   (c) $0{,}6$
   (d) $1{,}59$
   (e) $12{,}27\overline{7}$
3. Show that the decimal $3{,}2\dot{1}\dot{8}$ is a rational number.
4. Showing all working, express $0{,}7\dot{8}$ as a fraction $\frac{a}{b}$ where $a, b \in Z$.

Chapter 4

Exponentials - Grade 10

4.1 Introduction

In this chapter, you will learn about the short-cuts to writing $2 \times 2 \times 2 \times 2$. This is known as writing a number in *exponential notation*.

4.2 Definition

Exponential notation is a short way of writing the same number multiplied by itself many times. For example, instead of $5 \times 5 \times 5$, we write 5^3 to show that the number 5 is multiplied by itself 3 times and we say "5 to the power of 3". Likewise 5^2 is 5×5 and 3^5 is $3 \times 3 \times 3 \times 3 \times 3$. We will now have a closer look at writing numbers using exponential notation.

Definition: Exponential Notation
Exponential notation means a number written like

$$a^n$$

when n is an integer and a can be any real number. a is called the *base* and n is called the *exponent*.

The nth power of a is defined as:

$$a^n = 1 \times a \times a \times \ldots \times a \qquad (n \text{ times}) \tag{4.1}$$

with a multiplied by itself n times.

We can also define what it means if we have a negative index, $-n$. Then,

$$a^{-n} = 1 \div a \div a \div \ldots \div a \qquad (n \text{ times}) \tag{4.2}$$

Important: Exponentials

If n is an even integer, then a^n will always be positive for any non-zero real number a. For example, although -2 is negative, $(-2)^2 = 1 \times -2 \times -2 = 4$ is positive and so is $(-2)^{-2} = 1 \div -2 \div -2 = \frac{1}{4}$.

4.3 Laws of Exponents

There are several laws we can use to make working with exponential numbers easier. Some of these laws might have been seen in earlier grades, but we will list all the laws here for easy reference, but we will explain each law in detail, so that you can understand them, and not only remember them.

$$a^0 = 1 \quad (4.3)$$
$$a^m \times a^n = a^{m+n} \quad (4.4)$$
$$a^{-n} = \frac{1}{a^n} \quad (4.5)$$
$$a^m \div a^n = a^{m-n} \quad (4.6)$$
$$(ab)^n = a^n b^n \quad (4.7)$$
$$(a^m)^n = a^{mn} \quad (4.8)$$

4.3.1 Exponential Law 1: $a^0 = 1$

Our definition of exponential notation shows that

$$a^0 = 1, (a \neq 0) \quad (4.9)$$

For example, $x^0 = 1$ and $(1\ 000\ 000)^0 = 1$.

Exercise: Application using Exponential Law 1: $a^0 = 1, (a \neq 0)$

1. $16^0 = 1$
2. $16a^0 = 16$
3. $(16 + a)^0 = 1$
4. $(-16)^0 = 1$
5. $-16^0 = -1$

4.3.2 Exponential Law 2: $a^m \times a^n = a^{m+n}$

Our definition of exponential notation shows that

$$\begin{aligned} a^m \times a^n &= 1 \times a \times \ldots \times a \quad (m \text{ times}) \\ &\quad \times 1 \times a \times \ldots \times a \quad (n \text{ times}) \\ &= 1 \times a \times \ldots \times a \quad (m+n \text{ times}) \\ &= a^{m+n} \end{aligned} \quad (4.10)$$

For example,

$$\begin{aligned} 2^7 \times 2^3 &= (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) \\ &= 2^{10} \\ &= 2^{7+3} \end{aligned}$$

This simple law is the reason why exponentials were originally invented. In the days before calculators, all multiplication had to be done by hand with a pencil and a pad of paper. Multiplication takes a very long time to do and is very tedious. Adding numbers however, is very easy and quick to do. If you look at what this law is saying you will realise that it means that adding the exponents of two exponential numbers (of the same base) is the same as multiplying the two numbers together. This meant that for certain numbers, there was no need to actually multiply the numbers together in order to find out what their multiple was. This saved mathematicians a lot of time, which they could use to do something more productive.

Exercise: Application using Exponential Law 2: $a^m \times a^n = a^{m+n}$

1. $x^2.x^5 = x^7$
2. $2x^3y \times 5x^2y^7 = 10x^5y^8$
3. $2^3.2^4 = 2^7$ *[Take note that the base (2) stays the same.]*
4. $3 \times 3^{2a} \times 3^2 = 3^{2a+3}$

4.3.3 Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$

Our definition of exponential notation for a negative exponent shows that

$$\begin{aligned} a^{-n} &= 1 \div a \div \ldots \div a \qquad (n \text{ times}) \\ &= \frac{1}{1 \times a \times \cdots \times a} \qquad (n \text{ times}) \\ &= \frac{1}{a^n} \end{aligned} \tag{4.11}$$

This means that a minus sign in the exponent is just another way of writing that the whole exponential number is to be divided instead of multiplied.

For example,

$$\begin{aligned} 2^{-7} &= \frac{1}{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2} \\ &= \frac{1}{2^7} \end{aligned}$$

Exercise: Application using Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$

1. $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
2. $\frac{2^{-2}}{3^2} = \frac{1}{2^2.3^2} = \frac{1}{36}$
3. $(\frac{2}{3})^{-3} = (\frac{3}{2})^3 = \frac{27}{8}$

4. $\frac{m}{n^{-4}} = mn^4$
5. $\frac{a^{-3}.x^4}{a^5.x^{-2}} = \frac{x^4.x^2}{a^3.a^5} = \frac{x^6}{a^8}$

4.3.4 Exponential Law 4: $a^m \div a^n = a^{m-n}$

We already realised with law 3 that a minus sign is another way of saying that the exponential number is to be divided instead of multiplied. Law 4 is just a more general way of saying the same thing. We can get this law by just multiplying law 3 by a^m on both sides and using law 2.

$$\begin{aligned} \frac{a^m}{a^n} &= a^m a^{-n} \\ &= a^{m-n} \end{aligned} \tag{4.12}$$

For example,

$$\begin{aligned} 2^7 \div 2^3 &= \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} \\ &= 2 \times 2 \times 2 \times 2 \\ &= 2^4 \\ &= 2^{7-3} \end{aligned}$$

Exercise: Exponential Law 4: $a^m \div a^n = a^{m-n}$

1. $\frac{a^6}{a^2} = a^{6-2} = a^4$
2. $\frac{3^2}{3^6} = 3^{2-6} = 3^{-4} = \frac{1}{3^4}$ *[Always give final answer with positive index]*
3. $\frac{32a^2}{4a^8} = 8a^{-6} = \frac{8}{a^6}$
4. $\frac{a^{3x}}{a^4} = a^{3x-4}$

4.3.5 Exponential Law 5: $(ab)^n = a^n b^n$

The order in which two real numbers are multiplied together does not matter. Therefore,

$$\begin{aligned} (ab)^n &= a \times b \times a \times b \times \ldots \times a \times b \qquad (n \text{ times}) \\ &= a \times a \times \ldots \times a \qquad (n \text{ times}) \\ &\quad \times b \times b \times \ldots \times b \qquad (n \text{ times}) \\ &= a^n b^n \end{aligned} \tag{4.13}$$

For example,

$$\begin{aligned} (2 \cdot 3)^4 &= (2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3) \\ &= (2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3) \\ &= (2^4) \times (3^4) \\ &= 2^4 3^4 \end{aligned}$$

Exercise: Exponential Law 5: $(ab)^n = a^n b^n$

1. $(2x^2y)^3 = 2^3x^{2\times 3}y^5 = 8x^6y^5$
2. $(\frac{7a}{b^3})^2 = \frac{49a^2}{b^6}$
3. $(5a^{n-4})^3 = 125a^{3n-12}$

4.3.6 Exponential Law 6: $(a^m)^n = a^{mn}$

We can find the exponential of an exponential just as well as we can for a number. After all, an exponential number is a real number.

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times \ldots \times a^m \qquad (n \text{ times}) \\ &= a \times a \times \ldots \times a \qquad (m \times n \text{ times}) \\ &= a^{mn}\end{aligned} \tag{4.14}$$

For example,

$$\begin{aligned}(2^2)^3 &= (2^2) \times (2^2) \times (2^2) \\ &= (2\times 2)\times(2\times 2)\times(2\times 2) \\ &= (2^6) \\ &= 2^{(2\times 3)}\end{aligned}$$

Exercise: Exponential Law 6: $(a^m)^n = a^{mn}$

1. $(x^3)^4 = x^{12}$
2. $[(a^4)^3]^2 = a^{24}$
3. $(3^{n+3})^2 = 3^{2n+6}$

Worked Example 1: Simlifying indices

Question: Simplify: $\frac{5^{2x-1}.9^{x-2}}{15^{2x-3}}$
Answer
Step 1 : Factorise all bases into prime factors:

$$\begin{aligned} &= \frac{5^{2x-1}.(3^2)^{x-2}}{(5.3)^{2x-3}} \\ &= \frac{5^{2x-1}.3^{2x-4}}{5^{2x-3}.3^{2x-3}}\end{aligned}$$

Step 2 : Add and subtract the indices of the sam bases as per laws 2 and 4:

$$= 5^{2x-1-2x-3}.3^{2x-4-2x+3}$$
$$= 5^{2}.3^{-1}$$

Step 3 : Write simplified answer with positive indices:

$$= \frac{25}{3}$$

Activity :: Investigation : Exponential Numbers
Match the answers to the questions, by filling in the correct answer into the **Answer** column. Possible answers are: $\frac{3}{2}$, 1, -1, -3, 8.

Question	Answer
2^3	
7^{3-3}	
$(\frac{2}{3})^{-1}$	
8^{7-6}	
$(-3)^{-1}$	
$(-1)^{23}$	

4.4 End of Chapter Exercises

1. Simplify as far as possible:

   (a) 302^0 (b) 1^0 (c) $(xyz)^0$ (d) $[(3x^4y^7z^{12})^5(-5x^9y^3z^4)^2]^0$
   (e) $(2x)^3$ (f) $(-2x)^3$ (g) $(2x)^4$ (h) $(-2x)^4$

2. Simplify without using a calculator. Leave your answers with positive exponents.

   (a) $\frac{3x^{-3}}{(3x)^2}$

   (b) $5x^0 + 8^{-2} - (\frac{1}{2})^{-2}.1^x$

   (c) $\frac{5^{b-3}}{5^{b+1}}$

3. Simplify, showing all steps:

   (a) $\frac{2^{a-2}.3^{a+3}}{6^a}$

   (b) $\frac{a^{2m+n+p}}{a^{m+n+p}.a^m}$

   (c) $\frac{3^n.9^{n-3}}{27^{n-1}}$

   (d) $(\frac{2x^{2a}}{y^{-b}})^3$

(e) $\frac{2^{3x-1}.8^{x+1}}{4^{2x-2}}$

(f) $\frac{6^{2x}.11^{2x}}{22^{2x-1}.3^{2x}}$

4. Simplify, without using a calculator:

   (a) $\frac{(-3)^{-3}.(-3)^{2}}{(-3)^{-4}}$.

   (b) $(3^{-1}+2^{-1})^{-1}$

   (c) $\frac{9^{n-1}.27^{3-2n}}{81^{2-n}}$

   (d) $\frac{2^{3n+2}.8^{n-3}}{4^{3n-2}}$

Chapter 5

Estimating Surds - Grade 10

5.1 Introduction

You should known by now what the nth root of a number means. If the nth root of a number cannot be simplified to a rational number, we call it a *surd*. For example, $\sqrt{2}$ and $\sqrt[3]{6}$ are surds, but $\sqrt{4}$ is not a surd because it can be simplified to the rational number 2.

In this chapter we will only look at surds that look like $\sqrt[n]{a}$, where a is any positive number, for example $\sqrt{7}$ or $\sqrt[3]{5}$. It is very common for n to be 2, so we usually do not write $\sqrt[2]{a}$. Instead we write the surd as just $\sqrt{a}$, which is much easier to read.

It is sometimes useful to know the approximate value of a surd without having to use a calculator. For example, we want to be able to guess where a surd like $\sqrt{3}$ is on the number line. So how do we know where surds lie on the number line? From a calculator we know that $\sqrt{3}$ is equal to $1{,}73205...$. It is easy to see that $\sqrt{3}$ is above 1 and below 2. But to see this for other surds like $\sqrt{18}$ without using a calculator, you must first understand the following fact:

If a and b are positive whole numbers, and $a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$. (Challenge: Can you explain why?)

If you don't believe this fact, check it for a few numbers to convince yourself it is true.

How do we use this fact to help us guess what $\sqrt{18}$ is? Well, you can easily see that $18 < 25$? Using our rule, we also know that $\sqrt{18} < \sqrt{25}$. But we know that $5^2 = 25$ so that $\sqrt{25} = 5$. Now it is easy to simplify to get $\sqrt{18} < 5$. Now we have a better idea of what $\sqrt{18}$ is.

Now we know that $\sqrt{18}$ is less than 5, but this is only half the story. We can use the same trick again, but this time with 18 on the right-hand side. You will agree that $16 < 18$. Using our rule again, we also know that $\sqrt{16} < \sqrt{18}$. But we know that 16 is a perfect square, so we can simplify $\sqrt{16}$ to 4, and so we get $4 < \sqrt{18}$!

Can you see now that we now have shown that $\sqrt{18}$ is between 4 and 5? If we check on our calculator, we can see that $\sqrt{18} = 4{,}1231...$, and we see that our idea was right! You will notice that our idea used perfect squares that were close to the number 18. We found the closest perfect square underneath 18, which was $4^2 = 16$, and the closest perfect square above 18, which was $5^2 = 25$. Here is a quick summary of what a perfect square or cube is:

A perfect square is the number obtained when an integer is squared. For example, 9 is a perfect square since $3^2 = 9$. Similarly, a perfect cube is a number which is the cube of an integer. For example, 27 is a perfect cube, because $3^3 = 27$.

To make it easier to use our idea, we will create a list of some of the perfect squares and perfect cubes. The list is shown in Table 5.1.

Table 5.1: Some perfect squares and perfect cubes

Integer	Perfect Square	Perfect Cube
0	0	0
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000

Similarly, when given the surd $\sqrt[3]{52}$ you should be able to tell that it lies somewhere between 3 and 4, because $\sqrt[3]{27} = 3$ and $\sqrt[3]{64} = 4$ and 52 is between 27 and 64. In fact $\sqrt[3]{52} = 3{,}73\ldots$ which is indeed between 3 and 4.

5.2 Drawing Surds on the Number Line (Optional)

How can we accurately draw a surd like $\sqrt{5}$ on the number line? Well, we *could* use a calculator to find $\sqrt{5} = 2{,}2360...$ and measure the distance along the number line using a ruler. But for some surds, there is a much easier way.

Let us call the surd we are working with $\sqrt{a}$. Sometimes, we can write a as the sum of two perfect squares, so $a = b^2 + c^2$. We know from Pythagoras' theorem that $\sqrt{a} = \sqrt{b^2 + c^2}$ is the length of the hypotenuse of a triangle that has sides that have lengths of b and c. So if we draw a triangle on the number line with sides of length b and c, we can use a compass to draw a circle from the top of the hypotenuse down to the number line. The intersection will be the point $\sqrt{b}$ on the number line!

Not all numbers can be written as the sum of two squares. See if you can find a pattern of the numbers that can.

Worked Example 2: Estimating Surds

Question: Find the two consecutive integers such that $\sqrt{26}$ lies between them. (Remember that consecutive numbers that are two numbers one after the other, like 5 and 6 or 8 and 9.)

Answer

Step 1 : From the table find the largest perfect square below 26

This is $5^2 = 25$. Therefore $5 < \sqrt{26}$.

Step 2 : From the table find smallest perfect square above 26
This is $6^2 = 36$. Therefore $\sqrt{26} < 6$.
Step 3 : Put the inequalities together
Our answer is $5 < \sqrt{26} < 6$.

Worked Example 3: Estimating Surds

Question: $\sqrt[3]{49}$ lies between: (a) 1 and 2 (b) 2 and 3 (c) 3 and 4 (d) 4 and 5
Answer
Step 1 : Consider (a) as the solution
If $1 < \sqrt[3]{49} < 2$ then cubing all terms gives $1 < 49 < 2^3$. Simplifying gives $1 < 49 < 8$ which is false. So $\sqrt[3]{49}$ does not lie between 1 and 2.
Step 2 : Consider (b) as the solution
If $2 < \sqrt[3]{49} < 3$ then cubing all terms gives $2^3 < 49 < 3^3$. Simplifying gives $8 < 49 < 27$ which is false. So $\sqrt[3]{49}$ does not lie between 2 and 3.
Step 3 : Consider (c) as the solution
If $3 < \sqrt[3]{49} < 4$ then cubing all terms gives $3^3 < 49 < 4^3$. Simplifying gives $27 < 49 < 64$ which is true. So $\sqrt[3]{49}$ lies between 3 and 4.

5.3 End of Chapter Excercises

1.	$\sqrt{5}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
2.	$\sqrt{10}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
3.	$\sqrt{20}$ lies between	(a) 2 and 3	(b) 3 and 4	(c) 4 and 5	(d) 5 and 6
4.	$\sqrt{30}$ lies between	(a) 3 and 4	(b) 4 and 5	(c) 5 and 6	(d) 6 and 7
5.	$\sqrt[3]{5}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
6.	$\sqrt[3]{10}$ lies between	(a) 1 and 2	(b) 2 and 3	(c) 3 and 4	(d) 4 and 5
7.	$\sqrt[3]{20}$ lies between	(a) 2 and 3	(b) 3 and 4	(c) 4 and 5	(d) 5 and 6
8.	$\sqrt[3]{30}$ lies between	(a) 3 and 4	(b) 4 and 5	(c) 5 and 6	(d) 6 and 7

Chapter 6

Irrational Numbers and Rounding Off - Grade 10

6.1 Introduction

You have seen that repeating decimals may take a lot of paper and ink to write out. Not only is that impossible, but writing numbers out to many decimal places or *a high accuracy* is very inconvenient and rarely gives better answers. For this reason we often estimate the number to a certain number of decimal places or to a given number of *significant figures*, which is even better.

6.2 Irrational Numbers

Activity :: Investigation : Irrational Numbers
Which of the following cannot be written as a rational number?
Remember: A rational number is a fraction with numerator and denominator as integers. Terminating decimal numbers or repeating decimal numbers are rational.

1. $\pi = 3{,}14159265358979323846264338327950288419716939937510\ldots$
2. 1,4
3. $1{,}618\,033\,989\ldots$
4. 100

Irrational numbers are numbers that cannot be written as a rational number. You should know that a rational number can be written as a fraction with the numerator and denominator as integers. This means that any number that is *not* a terminating decimal number or a repeating decimal number are irrational. Examples of irrational numbers are:

$$\sqrt{2}, \quad \sqrt{3}, \quad \sqrt[3]{4}, \quad \pi,$$
$$\frac{1+\sqrt{5}}{2} \approx 1{,}618\,033\,989$$

! **Important:** When irrational numbers are written in decimal form, they go on forever and there is no repeated pattern of digits.

Important: Irrational Numbers

If you are asked to identify whether a number is rational or irrational, first write the number in decimal form. If the number is terminated then it is rational. If it goes on forever, then look for a repeated pattern of digits. If there is no repeated pattern, then the number is irrational.

When you write irrational numbers in decimal form, you may (if you have a lot of time and paper!) continue writing them for many, many decimal places. However, this is not convenient and it is often necessary to round off.

6.3 Rounding Off

Rounding off or approximating a decimal number to a given number of decimal places is the quickest way to approximate a number. For example, if you wanted to round-off $2{,}6525272$ to three decimal places then you would first count three places after the decimal.

$$2{,}652|5272$$

All numbers to the right of | are ignored after you determine whether the number in the third decimal place must be rounded up or rounded down. You *round up* the final digit if the first digit after the | was greater or equal to 5 and *round down* (leave the digit alone) otherwise.

So, since the first digit after the | is a 5, we must round up the digit in the third decimal place to a 3 and the final answer of $2{,}6525272$ rounded to three decimal places is

$$2{,}653$$

Worked Example 4: Rounding-Off

Question: Round-off the following numbers to the indicated number of decimal places:

1. $\frac{120}{99} = 1{,}21212121\dot{2}\dot{1}$ to 3 decimal places
2. $\pi = 3{,}141592654\ldots$ to 4 decimal places
3. $\sqrt{3} = 1{,}7320508\ldots$ to 4 decimal places

Answer

Step 1 : Determine the last digit that is kept and mark the cut-off point with |.

1. $\frac{120}{99} = 1{,}212|121212\dot{1}\dot{2}$
2. $\pi = 3{,}1415|92654\ldots$
3. $\sqrt{3} = 1{,}7320|508\ldots$

Step 2 : Determine whether the last digit is rounded up or down.

1. The last digit of $\frac{120}{99} = 1{,}212|121212\dot{1}\dot{2}$ must be rounded-down.
2. The last digit of $\pi = 3{,}1415|92654\ldots$ must be rounded-up.
3. The last digit of $\sqrt{3} = 1{,}7320|508\ldots$ must be rounded-up.

Step 3 : Write the final answer.

1. $\frac{120}{99} = 1{,}212$ rounded to 3 decimal places
2. $\pi = 3{,}1416$ rounded to 4 decimal places
3. $\sqrt{3} = 1{,}7321$ rounded to 4 decimal places

6.4 End of Chapter Exercises

1. Write the following rational numbers to 2 decimal places:
   (a) $\frac{1}{2}$
   (b) 1
   (c) $0{,}11111\overline{1}$
   (d) $0{,}99999\overline{1}$
2. Write the following irrational numbers to 2 decimal places:
   (a) $3{,}141592654\ldots$
   (b) $1{,}618\,033\,989\ldots$
   (c) $1{,}41421356\ldots$
   (d) $2{,}71828182845904523536\ldots$
3. Use your calculator and write the following irrational numbers to 3 decimal places:
   (a) $\sqrt{2}$
   (b) $\sqrt{3}$
   (c) $\sqrt{5}$
   (d) $\sqrt{6}$
4. Use your calculator (where necessary) and write the following irrational numbers to 5 decimal places:
   (a) $\sqrt{8}$
   (b) $\sqrt{768}$
   (c) $\sqrt{100}$
   (d) $\sqrt{0{,}49}$
   (e) $\sqrt{0{,}0016}$
   (f) $\sqrt{0{,}25}$
   (g) $\sqrt{36}$
   (h) $\sqrt{1960}$
   (i) $\sqrt{0{,}0036}$
   (j) $-8\sqrt{0{,}04}$
   (k) $5\sqrt{80}$
5. Write the following irrational numbers to 3 decimal places and then write them as a rational number to get an approximation to the irrational number. For example, $\sqrt{3} = 1{,}73205\ldots$. To 3 decimal places, $\sqrt{3} = 1{,}732$. $1{,}732 = 1\frac{732}{1000} = 1\frac{183}{250}$. Therefore, $\sqrt{3}$ is approximately $1\frac{183}{250}$.
   (a) $3{,}141592654\ldots$
   (b) $1{,}618\,033\,989\ldots$
   (c) $1{,}41421356\ldots$
   (d) $2{,}71828182845904523536\ldots$

Chapter 7

Number Patterns - Grade 10

In earlier grades you saw patterns in the form of pictures and numbers. In this chapter we learn more about the mathematics of patterns.Patterns are recognisable regularities in situations such as in nature, shapes, events, sets of numbers. For example, spirals on a pineapple, snowflakes, geometric designs on quilts or tiles, the number sequence 0, 4, 8, 12, 16,....

Activity :: Investigation : Patterns
Can you spot any patterns in the following lists of numbers?

1. 2; 4; 6; 8; 10; . . .
2. 1; 2; 4; 7; 11; . . .
3. 1; 4; 9; 16; 25; . . .
4. 5; 10; 20; 40; 80; . . .

7.1 Common Number Patterns

Numbers can have interesting patterns. Here we list the most common patterns and how they are made.

Examples:

1. $1, 4, 7, 10, 13, 16, 19, 22, 25, ...$

   This sequence has a difference of 3 between each number. The pattern is continued by adding 3 to the last number each time.

2. $3, 8, 13, 18, 23, 28, 33, 38, ...$

   This sequence has a difference of 5 between each number. The pattern is continued by adding 5 to the last number each time.

3. $2, 4, 8, 16, 32, 64, 128, 256, ...$

   This sequence has a factor of 2 between each number. The pattern is continued by multiplying the last number by 2 each time.

4. $3, 9, 27, 81, 243, 729, 2187, ...$

   This sequence has a factor of 3 between each number. The pattern is continued by multiplying the last number by 3 each time.

7.1.1 Special Sequences

Triangular Numbers

$1, 3, 6, 10, 15, 21, 28, 36, 45, ...$

This sequence is generated from a pattern of dots which form a triangle. By adding another row of dots and counting all the dots we can find the next number of the sequence.

Square Numbers

$1, 4, 9, 16, 25, 36, 49, 64, 81, ...$

The next number is made by squaring where it is in the pattern. The second number is 2 squared (2^2 or 2×2) The seventh number is 7 squared (7^2 or 7×7) etc

Cube Numbers

$1, 8, 27, 64, 125, 216, 343, 512, 729, ...$

The next number is made by cubing where it is in the pattern. The second number is 2 cubed (2^3 or $2 \times 2 \times 2$) The seventh number is 7 cubed (7^3 or $7 \times 7 \times 7$) etc

Fibonacci Numbers

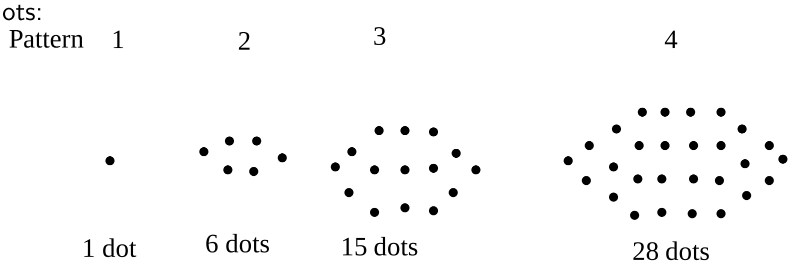
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$

The next number is found by adding the two numbers before it together. The 2 is found by adding the two numbers in front of it $(1 + 1)$ The 21 is found by adding the two numbers in front of it $(8 + 13)$ The next number in the sequence above would be 55 $(21 + 34)$

Can you figure out the next few numbers?

7.2 Make your own Number Patterns

You can make your own number patterns using coins or matchsticks. Here is an example using dots:



How many dots would you need for pattern 5 ? Can you make a formula that will tell you how many coins are needed for any size pattern? For example if the pattern 20? The formula may look something like

$$dots = pattern \times pattern + ...$$

Worked Example 5: Study Table

Question: Say you and 3 friends decide to study for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you and would like to sit at your table and help you study. Naturally, you move another table and add it to the existing one. Now six of you sit at the table. Another two of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you can sit comfortably.

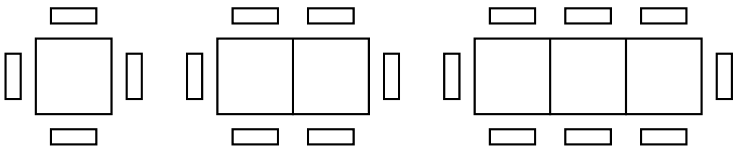


Figure 7.1: Two more people can be seated for each table added.

Examine how the number of people sitting is related to the number of tables.

Answer

Step 1 : Tabulate a few terms to see if there is a pattern

Number of Tables, n	**Number of people seated**
1	$4 = 4$
2	$4 + 2 = 6$
3	$4 + 2 + 2 = 8$
4	$4 + 2 + 2 + 2 = 10$
$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \ldots + 2$

Step 2 : Describe the pattern

We can see for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two the whole time. Thus, for each table added, the number of persons increases by two.

7.3 Notation

A sequence does not have to follow a pattern but when it does we can often write down a formula to calculate the n^{th}-term, a_n. In the sequence

$$1; 4; 9; 16; 25; \ldots$$

where the sequence consists of the squares of integers, the formula for the n^{th}-term is

$$a_n = n^2 \quad (7.1)$$

You can check this by looking at:

$$\begin{aligned} a_1 &= 1^2 = 1 \\ a_2 &= 2^2 = 4 \\ a_3 &= 3^2 = 9 \\ a_4 &= 4^2 = 16 \\ a_5 &= 5^2 = 25 \\ \ldots \end{aligned}$$

Therefore, using (7.1), we can generate a pattern, namely squares of integers.

Worked Example 6: Study Table continued ....

Question: As before, you and 3 friends are studying for Maths, and you are seated at a square table. A few minutes later, 2 other friends join you move another table and add it to the existing one. Now six of you sit at the table. Another two of your friends join your table, and you take a third table and add it to the existing tables. Now 8 of you sit comfortably as illustrated:

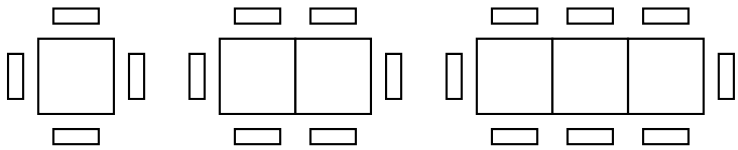


Figure 7.2: Two more people can be seated for each table added.

Find the expression for the number of people seated at n tables. Then, use the general formula to determine how many people can sit around 12 tables and how many tables are needed for 20 people.

Answer

Step 1 : Tabulate a few terms to see if there is a pattern

Number of Tables, n	Number of people seated	Formula
1	$4 = 4$	$= 4 + 2 \cdot (0)$
2	$4 + 2 = 6$	$= 4 + 2 \cdot (1)$
3	$4 + 2 + 2 = 8$	$= 4 + 2 \cdot (2)$
4	$4 + 2 + 2 + 2 = 10$	$= 4 + 2 \cdot (3)$
$\vdots$	$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \ldots + 2$	$= 4 + 2 \cdot (n - 1)$

Step 2 : Describe the pattern

The number of people seated at n tables is:

$$a_n = 4 + 2 \cdot (n - 1)$$

Step 3 : Calculate the 12^{th} term

Using the general formula (36.1) and considering the example from the previous section, how many people can sit around, say, 12 tables? We are looking for a_{12}, that is, where $n = 12$:

$$\begin{aligned} a_n &= a_1 + d \cdot (n - 1) \\ a_{12} &= 4 + 2 \cdot (12 - 1) \\ &= 4 + 2(11) \\ &= 4 + 22 \\ &= 26 \end{aligned}$$

Step 4 : Calculate the number of terms if $a_n = 20$

$$\begin{aligned} a_n &= a_1 + d \cdot (n-1) \\ 20 &= 4 + 2 \cdot (n-1) \\ 20 - 4 &= 2 \cdot (n-1) \\ 16 \div 2 &= n - 1 \\ 8 + 1 &= n \\ n &= 9 \end{aligned}$$

Step 5 : Final Answer

26 people can be seated at 12 tables and 9 tables are needed to seat 20 people.

It is also important to note the difference between n and a_n. n can be compared to a place holder, while a_n is the value at the place "held" by n. Like our "Study Table"-example above, the first table (Table 1) holds 4 people. Thus, at place $n = 1$, the value of $a_1 = 4$, and so on:

n	1	2	3	4	...
a_n	4	6	8	10	...

Activity :: Investigation : General Formula

1. Find the general formula for the following sequences and then find a_{10}, a_{50} and a_{100}:
   (a) $2, 5, 8, 11, 14, \ldots$
   (b) $0, 4, 8, 12, 16, \ldots$
   (c) $2, -1, -4, -7, -10, \ldots$
2. The general term has been given for each sequence below. Work out the missing terms.
   (a) $0; 3; ...; 15; 24$ $\quad n^2 - 1$
   (b) $3; 2; 1; 0; ...; 2$ $\quad -n + 4$
   (c) $11; ...; 7; ...; 3$ $\quad -13 + 2n$

7.3.1 Patterns and Conjecture

In mathematics, a conjecture is a mathematical statement which appears to be true, but has not been formally proven to be true under the rules of mathematics. Other words that have a similar in meaning to conjecture are: hypothesis, theory, assumption and premise.

For example: Make a **conjecture** about the next number based on the pattern $2; 6; 11; 17 : ...$

The numbers increase by 4, 5, and 6.

Conjecture: The next number will increase by 7. So, it will be $17 + 7$ or 24.

Worked Example 7: Number patterns

Question: Consider the following pattern.

$$\begin{aligned} 1^2+1 &= 2^2-2 \\ 2^2+2 &= 3^2-3 \\ 3^2+3 &= 4^2-4 \\ 4^2+4 &= 5^2-5 \end{aligned}$$

1. Add another two rows to the end of the pattern.
2. Make a conjecture about this pattern. Write your conjecture in words.
3. Generalise your conjecture for this pattern (in other words, write your conjecture algebraically).
4. Prove that your conjecture is true.

Answer
Step 1 : The next two rows

$$5^2+5=6^2-6$$
$$6^2+6=7^2-7$$

Step 2 : Conjecture
Squaring a number and adding the same number gives the same result as squaring the next number and subtracting that number.
Step 3 : Generalise
We have chosen to use x here. You could choose any letter to generalise the pattern.

$$x^2+x=(x+1)^2-(x+1)$$

Step 4 : Proof

$$Left\ side:\ x^2+x$$

$$Right\ side:\ (x+1)^2-(x+1)$$

$$\begin{aligned} Right\ side &= x^2+2x+1-x-1 \\ &= x^2+x \\ &= left\ side \end{aligned}$$
$$Therefore\ x^2+x=(x+1)^2-(x+1)$$

7.4 Exercises

1. Find the n^{th} term for: $3, 7, 11, 15, \ldots$
2. Find the general term of the following sequences:

(a) $-2,1,4,7,\ldots$
(b) $11,15,19,23,\ldots$
(c) $x-1,2x+5,5x+1,\ldots$
(d) sequence with $a_3 = 7$ and $a_8 = 15$
(e) sequence with $a_4 = -8$ and $a_{10} = 10$

3. The seating in a section of a sports stadium can be arranged so the first row has 15 seats, the second row has 19 seats, the third row has 23 seats and so on. Calculate how many seats are in the row 25.

4. Consider the following pattern:

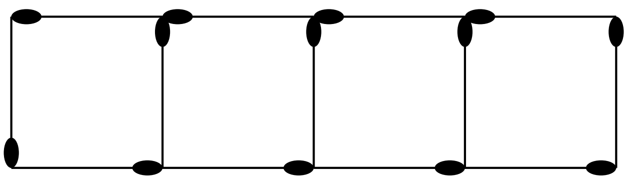
$$\begin{aligned} 2^2+2 &= 3^2-3 \\ 3^2+3 &= 4^2-4 \\ 4^2+4 &= 5^2-5 \end{aligned}$$

(a) Add at least two more rows to the pattern and check whether or not the pattern continues to work.
(b) Describe in words any patterns that you have noticed.
(c) Try to generalise a rule using algebra i.e. find the general term for the pattern.
(d) Prove or disprove that this rule works for all values.

5. The profits of a small company for the last four years has been: R10 000, R15 000, R19 000 and R23 000. If the pattern continues, what is the expected profit in the 10 years (i.e. in the 14^{th} year of the company being in business)?

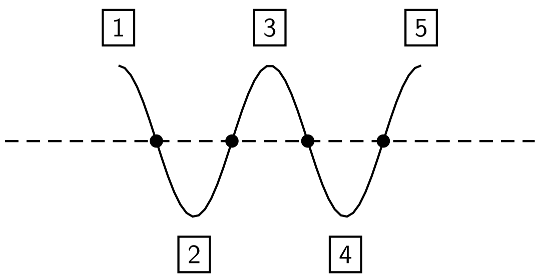
6. A single square is made from 4 matchsticks. Two squares in a row needs 7 matchsticks and 3 squares in a row needs 10 matchsticks. Determine:

(a) the first term
(b) the common difference
(c) the formula for the general term
(d) how many matchsticks are in a row of 25 squares



7. You would like to start saving some money, but because you have never tried to save money before, you have decided to start slowly. At the end of the first week you deposit R5 into your bank account. Then at the end of the second week you deposit R10 into your bank account. At the end of the third week you deposit R15. After how many weeks, do you deposit R50 into your bank account?

8. A horizontal line intersects a piece of string at four points and divides it into five parts, as shown below.



If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at four points, find the number of parts into which the string will be divided.

Chapter 8

Finance - Grade 10

8.1 Introduction

Should you ever find yourself stuck with a mathematics question on a television quiz show, you will probably wish you had remembered the how many even prime numbers there are between 1 and 100 for the sake of R1 000 000. And who does not want to be a millionaire, right?

Welcome to the Grade 10 Finance Chapter, where we apply maths skills to everyday financial situations that you are likely to face both now and along your journey to purchasing your first private jet.

If you master the techniques in this chapter, you will grasp the concept of *compound interest*, and how it can ruin your fortunes if you have credit card debt, or make you millions if you successfully invest your hard-earned money. You will also understand the effects of fluctuating exchange rates, and its impact on your spending power during your overseas holidays!

8.2 Foreign Exchange Rates

Is \$500 ("500 US dollars") per person per night a good deal on a hotel in New York City? The first question you will ask is "How much is that worth in Rands?". A quick call to the local bank or a search on the Internet (for example on `http://www.x-rates.com/`) for the Dollar/Rand exchange rate will give you a basis for assessing the price.

A foreign exchange rate is nothing more than the price of one currency in terms of another. For example, the exchange rate of 6,18 Rands/US Dollars means that \$1 costs R6,18. In other words, if you have \$1 you could sell it for R6,18 - or if you wanted \$1 you would have to pay R6,18 for it.

But what drives exchange rates, and what causes exchange rates to change? And how does this affect you anyway? This section looks at answering these questions.

8.2.1 How much is R1 really worth?

We can quote the price of a currency in terms of any other currency, but the US Dollar, British Pounds Sterling or even the Euro are often used as a market standard. You will notice that the financial news will report the South African Rand exchange rate in terms of these three major currencies.

So the South African Rand could be quoted on a certain date as 6,7040 ZAR per USD (i.e. \$1,00 costs R6,7040), or 12,2374 ZAR per GBP. So if I wanted to spend \$1 000 on a holiday in the United States of America, this would cost me R6 704,00; and if I wanted £1 000 for a weekend in London it would cost me R12 237,40.

This seems obvious, but let us see how we calculated that: The rate is given as ZAR per USD, or ZAR/USD such that \$1,00 buys R6,7040. Therefore, we need to multiply by 1 000 to get the

Table 8.1: Abbreviations and symbols for some common currencies.

Currency	Abbreviation	Symbol
South African Rand	ZAR	R
United States Dollar	USD	$
British Pounds Sterling	GBP	£
Euro	EUR	€

number of Rands per $1 000.

Mathematically,

$$\begin{aligned} \$1{,}00 &= \text{R}6{,}0740 \\ \therefore \quad 1\ 000 \times \$1{,}00 &= 1\ 000 \times \text{R}6{,}0740 \\ &= \text{R}6\ 074{,}00 \end{aligned}$$

as expected.

What if you have saved R10 000 for spending money for the same trip and you wanted to use this to buy USD? How much USD could you get for this? Our rate is in ZAR/USD but we want to know how many USD we can get for our ZAR. This is easy. We know how much $1,00 costs in terms of Rands.

$$\begin{aligned} \$1{,}00 &= \text{R}6{,}0740 \\ \therefore \quad \frac{\$1{,}00}{6{,}0740} &= \frac{\text{R}6{,}0740}{6{,}0740} \\ \$\frac{1{,}00}{6{,}0740} &= \text{R}1{,}00 \\ \text{R}1{,}00 &= \$\frac{1{,}00}{6{,}0740} \\ &= \$0{,}164636 \end{aligned}$$

As we can see, the final answer is simply the reciprocal of the ZAR/USD rate. Therefore, R10 000 will get:

$$\begin{aligned} \text{R}1{,}00 &= \$\frac{1{,}00}{6{,}0740} \\ \therefore \quad 10\ 000 \times \text{R}1{,}00 &= 10\ 000 \times \$\frac{1{,}00}{6{,}0740} \\ &= \$1\ 646{,}36 \end{aligned}$$

We can check the answer as follows:

$$\begin{aligned} \$1{,}00 &= \text{R}6{,}0740 \\ \therefore \quad 1\ 646{,}36 \times \$1{,}00 &= 1\ 646{,}36 \times \text{R}6{,}0740 \\ &= \text{R}10\ 000{,}00 \end{aligned}$$

Six of one and half a dozen of the other

So we have two different ways of expressing the same exchange rate: Rands per Dollar (ZAR/USD) and Dollar per Rands (USD/ZAR). Both exchange rates mean the same thing and express the value of one currency in terms of another. You can easily work out one from the other - they are just the reciprocals of the other.

If the South African Rand is our Domestic (or home) Currency, we call the ZAR/USD rate a "direct" rate, and we call a USD/ZAR rate an "indirect" rate.

In general, a direct rate is an exchange rate that is expressed as units of Home Currency per units of Foreign Currency, i.e., Domestic Currency / Foreign Currency.

The Rand exchange rates that we see on the news are usually expressed as Direct Rates, for example you might see:

Table 8.2: Examples of exchange rates

Currency Abbreviation	Exchange Rates
1 USD	R6,9556
1 GBP	R13,6628
1 EUR	R9,1954

The exchange rate is just the price of each of the Foreign Currencies (USD, GBP and EUR) in terms of our Domestic Currency, Rands.

An indirect rate is an exchange rate expressed as units of Foreign Currency per units of Home Currency, i.e. Foreign Currency / Domestic Currency

Defining exchange rates as direct or indirect depends on which currency is defined as the Domestic Currency. The Domestic Currency for an American investor would be USD which is the South African investor's Foreign Currency. So direct rates from the perspective of the American investor (USD/ZAR) would be the same as the indirect rate from the perspective of the South Africa investor.

Terminology

Since exchange rates are simple prices of currencies, movements in exchange rates means that the price or value of the currency changed. The price of petrol changes all the time, so does the price of gold, and currency prices also move up and down all the time.

If the Rand exchange rate moved from say R6,71 per USD to R6,50 per USD, what does this mean? Well, it means that $1 would now cost only R6,50 instead of R6,71. The Dollar is now cheaper to buy, and we say that the Dollar has depreciated (or weakened) against the Rand. Alternatively we could say that the Rand has appreciated (or strengthened) against the Dollar.

What if we were looking at indirect exchange rates, and the exchange rate moved from $0,149 per ZAR ($=\frac{1}{6,71}$) to $0,1538 per ZAR ($=\frac{1}{6,50}$).

Well now we can see that the R1,00 cost $0,149 at the start, and then cost $0,1538 at the end. The Rand has become more expensive (in terms of Dollars), and again we can say that the Rand has appreciated.

Regardless of which exchange rate is used, we still come to the same conclusions.

In general,

- for direct exchange rates, the home currency will appreciate (depreciate) if the exchange rate falls (rises)
- For indirect exchange rates, the home currency will appreciate (depreciate) if the exchange rate rises (falls)

As with just about everything in this chapter, do not get caught up in memorising these formulae - that is only going to get confusing. Think about what you have and what you want - and it should be quite clear how to get the correct answer.

Activity :: Discussion : Foreign Exchange Rates

In groups of 5, discuss:

1. Why might we need to know exchange rates?
2. What happens if one countries currency falls drastically vs another countries currency?

3. When might you use exchange rates?

8.2.2 Cross Currency Exchange Rates

We know that the exchange rates are the value of one currency expressed in terms of another currency, and we can quote exchange rates against any other currency. The Rand exchange rates we see on the news are usually expressed against the major currency, USD, GBP and EUR.

So if for example, the Rand exchange rates were given as 6,71 ZAR/USD and 12,71 ZAR/GBP, does this tell us anything about the exchange rate between USD and GBP?

Well I know that if \$1 will buy me R6,71, and if £1.00 will buy me R12,71, then surely the GBP is stronger than the USD because you will get more Rands for one unit of the currency, and we can work out the USD/GBP exchange rate as follows:

Before we plug in any numbers, how can we get a USD/GBP exchange rate from the ZAR/USD and ZAR/GBP exchange rates?

Well,

$$\text{USD/GBP} = \text{USD/ZAR} \times \text{ZAR/GBP}.$$

Note that the ZAR in the numerator will cancel out with the ZAR in the denominator, and we are left with the USD/GBP exchange rate.

Although we do not have the USD/ZAR exchange rate, we know that this is just the reciprocal of the ZAR/USD exchange rate.

$$\text{USD/ZAR} = \frac{1}{\text{ZAR/USD}}$$

Now plugging in the numbers, we get:

$$\begin{aligned}
\text{USD/GBP} &= \text{USD/ZAR} \times \text{ZAR/GBP} \\
&= \frac{1}{\text{ZAR/USD}} \times \text{ZAR/GBP} \\
&= \frac{1}{6{,}71} \times 12{,}71 \\
&= 1{,}894
\end{aligned}$$

Important: Sometimes you will see exchange rates in the real world that do not appear to work exactly like this. This is usually because some financial institutions add other costs to the exchange rates, which alter the results. However, if you could remove the effect of those extra costs, the numbers would balance again.

Worked Example 8: Cross Exchange Rates

Question: If \$1 = R 6,40, and £1 = R11,58 what is the \$/£ exchange rate (i.e. the number of US\$ per £)?

Answer

Step 1 : Determine what is given and what is required

The following are given:

- ZAR/USD rate = R6,40
- ZAR/GBP rate = R11,58

The following is required:

- USD/GBP rate

Step 2 : Determine how to approach the problem

We know that:

$$\text{USD/GBP} = \text{USD/ZAR} \times \text{ZAR/GBP}.$$

Step 3 : Solve the problem

$$\begin{aligned}
\text{USD/GBP} &= \text{USD/ZAR} \times \text{ZAR/GBP} \\
&= \frac{1}{\text{ZAR/USD}} \times \text{ZAR/GBP} \\
&= \frac{1}{6{,}40} \times 11{,}58 \\
&= 1{,}8094
\end{aligned}$$

Step 4 : Write the final answer

\$1,8094 can be bought for £1.

Activity :: Investigation : Cross Exchange Rates - Alternate Method

If \$1 = R 6,40, and £1 = R11,58 what is the \$/£ exchange rate (i.e. the number of US\$ per £)?

Overview of problem

You need the \$/£ exchange rate, in other words how many dollars must you pay for a pound. So you need £1. From the given information we know that it would cost you R11,58 to buy £1 and that \$ 1 = R6,40.

Use this information to:

1. calculate how much R1 is worth in \$.
2. calculate how much R11,58 is worth in \$.

Do you get the same answer as in the worked example?

8.2.3 Enrichment: Fluctuating exchange rates

If everyone wants to buy houses in a certain suburb, then house prices are going to go up - because the buyers will be competing to buy those houses. If there is a suburb where all residents want to move out, then there are lots of sellers and this will cause house prices in the area to fall - because the buyers would not have to struggle as much to find an eager seller.

This is all about supply and demand, which is a very important section in the study of Economics. You can think about this is many different contexts, like stamp-collecting for example. If there is a stamp that lots of people want (high demand) and few people own (low supply) then that stamp is going to be expensive.

And if you are starting to wonder why this is relevant - think about currencies. If you are going to visit London, then you have Rands but you need to "buy" Pounds. The exchange rate is the price you have to pay to buy those Pounds.

Think about a time where lots of South Africans are visiting the United Kingdom, and other South Africans are importing goods from the United Kingdom. That means there are lots of Rands (high supply) trying to buy Pounds. Pounds will start to become more expensive (compare this to the house price example at the start of this section if you are not convinced), and the exchange rate will change. In other words, for R1 000 you will get fewer Pounds than you would have before the exchange rate moved.

Another context which might be useful for you to understand this: consider what would happen if people in other countries felt that South Africa was becoming a great place to live, and that more people were wanting to invest in South Africa - whether in properties, businesses - or just buying more goods from South Africa. There would be a greater demand for Rands - and the "price of the Rand" would go up. In other words, people would need to use more Dollars, or Pounds, or Euros ... to buy the same amount of Rands. This is seen as a movement in exchange rates.

Although it really does come down to supply and demand, it is interesting to think about what factors might affect the supply (people wanting to "sell" a particular currency) and the demand (people trying to "buy" another currency). This is covered in detail in the study of Economics, but let us look at some of the basic issues here.

There are various factors affect exchange rates, some of which have more economic rationale than others:

- economic factors (such as inflation figures, interest rates, trade deficit information, monetary policy and fiscal policy)
- political factors (such as uncertain political environment, or political unrest)
- market sentiments and market behaviour (for example if foreign exchange markets perceived a currency to be overvalued and starting selling the currency, this would cause the currency to fall in value - a self fulfilling expectation).

?

Exercise: Foreign Exchange

1. I want to buy an IPOD that costs $£100$, with the exchange rate currently at $£1 = R14$. I believe the exchange rate will reach $R12$ in a month.
   (a) How much will the MP3 player cost in Rands, if I buy it now?
   (b) How much will I save if the exchange rate drops to $R12$?
   (c) How much will I lose if the exchange rate moves to $R15$?
2. Study the following exchange rate table:

Country	Currency	Exchange Rate
United Kingdom (UK)	Pounds(£)	$R14{,}13$
United States (USA)	Dollars ($)	$R7{,}04$

   (a) In South Africa the cost of a new Honda Civic is $R173\,400$. In England the same vehicle costs $£12\,200$ and in the USA \$ $21\,900$. In which country is the car the cheapest if you compare it to the South African Rand ?
   (b) Sollie and Arinda are waiters in a South African Restaurant attracting many tourists from abroad. Sollie gets a $£6$ tip from a tourist and Arinda gets \$ 12. How many South African Rand did each one get ?

8.3 Being Interested in Interest

If you had R1 000, you could either keep it in your wallet, or deposit it in a bank account. If it stayed in your wallet, you could spend it any time you wanted. If the bank looked after it for you, then they could spend it, with the plan of making profit off it. The bank usually "pays" you to deposit it into an account, as a way of encouraging you to bank it with them, This payment is like a reward, which provides you with a reason to leave it with the bank for a while, rather than keeping the money in your wallet.

We call this reward "interest".

If you deposit money into a bank account, you are effectively lending money to the bank - and you can expect to receive interest in return. Similarly, if you borrow money from a bank (or from a department store, or a car dealership, for example) then you can expect to have to pay interest on the loan. That is the price of borrowing money.

The concept is simple, yet it is core to the world of finance. Accountants, actuaries and bankers, for example, could spend their entire working career dealing with the effects of interest on financial matters.

In this chapter you will be introduced to the concept of financial mathematics - and given the tools to cope with even advanced concepts and problems.

Important: Interest

The concepts in this chapter are simple - we are just looking at the same idea, but from many different angles. The best way to learn from this chapter is to do the examples yourself, as you work your way through. Do not just take our word for it!

8.4 Simple Interest

Definition: Simple Interest
Simple interest is where you earn interest on the initial amount that you invested, but not interest on interest.

As an easy example of simple interest, consider how much you will get by investing R1 000 for 1 year with a bank that pays you 5% simple interest. At the end of the year, you will get an interest of:

$$\begin{aligned} \text{Interest} &= \text{R1 000} \times 5\% \\ &= \text{R1 000} \times \frac{5}{100} \\ &= \text{R1 000} \times 0{,}05 \\ &= \text{R50} \end{aligned}$$

So, with an "opening balance" of R1 000 at the start of the year, your "closing balance" at the end of the year will therefore be:

$$\begin{aligned} \text{Closing Balance} &= \text{Opening Balance} + \text{Interest} \\ &= \text{R1 000} + \text{R50} \\ &= \text{R1 050} \end{aligned}$$

We sometimes call the opening balance in financial calculations *Principal*, which is abbreviated as P (R1 000 in the example). The interest rate is usually labelled i (5% in the example), and the interest amount (in Rand terms) is labelled I (R50 in the example).

So we can see that:

$$I = P \times i \qquad (8.1)$$

and

$$\begin{aligned} \text{Closing Balance} &= \text{Opening Balance} + \text{Interest} \\ &= P + I \\ &= P + (P \times i) \\ &= P(1 + i) \end{aligned}$$

This is how you calculate simple interest. It is not a complicated formula, which is just as well because you are going to see a lot of it!

Not Just One

You might be wondering to yourself:

1. how much interest will you be paid if you only leave the money in the account for 3 months, or

2. what if you leave it there for 3 years?

It is actually quite simple - which is why they call it **Simple Interest**.

1. Three months is 1/4 of a year, so you would only get 1/4 of a full year's interest, which is: $1/4 \times (P \times i)$. The closing balance would therefore be:

$$\begin{aligned}\text{Closing Balance} &= P + 1/4 \times (P \times i) \\ &= P(1 + (1/4)i)\end{aligned}$$

2. For 3 years, you would get three years' worth of interest, being: $3 \times (P \times i)$. The closing balance at the end of the three year period would be:

$$\begin{aligned}\text{Closing Balance} &= P + 3 \times (P \times i) \\ &= P \times (1 + (3)i)\end{aligned}$$

If you look carefully at the similarities between the two answers above, we can generalise the result. In other words, if you invest your money (P) in an account which pays a rate of interest (i) for a period of time (n years), then, using the symbol (A) for the Closing Balance:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n) \tag{8.2}$$

As we have seen, this works when n is a fraction of a year and also when n covers several years.

Important: Interest Calculation

Annual Rates means Yearly rates. and p.a.(per annum) = per year

Worked Example 9: Simple Interest

Question: If I deposit R1 000 into a special bank account which pays a Simple Interest of 7% for 3 years, how much will I get back at the end?

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R}1\ 000$
- interest rate, $i = 7\%$
- period of time, $n = 3$ years

We are required to find the closing balance (A).

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1+i\cdot n) \\ &= R1\ 000(1+3\times 7\%) \\ &= R1\ 210 \end{aligned}$$

Step 4 : Write the final answer

The closing balance after 3 years of saving R1 000 at an interest rate of 7% is R1 210.

Worked Example 10: Calculating n

Question: If I deposit R30 000 into a special bank account which pays a Simple Interest of 7.5% ,for how many years must I invest this amount to generate R45 000

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R}30\ 000$
- interest rate, $i = 7{,}5\%$
- closing balance, $A = \text{R}45\ 000$

We are required to find the number of years.

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance (A)} = P(1+i\cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{Closing Balance (A)} &= P(1+i\cdot n) \\ R45\ 000 &= R30\ 000(1+n\times 7{,}5\%) \\ (1+0{,}075\times n) &= \frac{45000}{30000} \\ 0{,}075\times n &= 1{,}5-1 \\ n &= \frac{0{,}5}{0{,}075} \\ n &= 6{,}6666667 \end{aligned}$$

Step 4 : Write the final answer

n has to be a whole number, therefore $n = 7$.

The period is 7 years for R30 000 to generate R45 000 at a simple interest rate of 7,5%.

8.4.1 Other Applications of the Simple Interest Formula

Worked Example 11: Hire-Purchase

Question: Troy is keen to buy an addisional hard drive for his laptop advertised for R 2 500 on the internet. There is an option of paying a 10% deposit then making 24 monthly payments using a hire-purchase agreement where interest is calculated at 7,5% p.a. simple interest. Calculate what Troy's monthly payments will be.

Answer

Step 1 : Determine what is given and what is required

A new opening balance is required, as the 10% deposit is paid in cash.

- 10% of R 2 500 = R250
- new opening balance, $P = \text{R2 500} - \text{R250} = \text{R2 250}$
- interest rate, $i = 7{,}5\% = 0{,}075\text{pa}$
- period of time, $n = 2\text{ years}$

We are required to find the closing balance (A) and then the montly payments.

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1 + i \cdot n) \\ &= R2\ 250(1 + 2 \times 7{,}5\%) \\ &= R2\ 587{,}50 \\ \text{Monthly payment} &= 2587{,}50 \div 24 \\ &= R107{,}81 \end{aligned}$$

Step 4 : Write the final answer

Troy's monthly payments = R 107,81

Worked Example 12: Depreciation

Question: Seven years ago, Tjad's drum kit cost him R12 500. It has now been valued at R2 300. What rate of simple depreciation does this represent ?

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R12 500}$
- period of time, $n = 7\text{ years}$
- closing balance, $A = \text{R2 300}$

We are required to find the rate(i).

Step 2 : Determine how to approach the problem

We know from (8.2) that:

$$\text{Closing Balance,(A)} = P(1 + i \cdot n)$$

Therefore, for **depreciation** the formula will change to:

$$\text{Closing Balance,(A)} = P(1 - i \cdot n)$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1 - i \cdot n) \\ R2\ 300 &= R12\ 500(1 - 7 \times i) \\ i &= 0{,}11657... \end{aligned}$$

Step 4 : Write the final answer
Therefore the rate of depreciation is $11{,}66\%$

?

Exercise: Simple Interest

1. An amount of R3 500 is invested in a savings account which pays simple interest at a rate of 7,5% per annum. Calculate the balance accumulated by the end of 2 years.
2. Calculate the simple interest for the following problems.
   (a) A loan of R300 at a rate of 8% for l year.
   (b) An investment of R225 at a rate of 12,5% for 6 years.
3. I made a deposit of R5 000 in the bank for my 5 year old son's 21st birthday. I have given him the amount of R 18 000 on his birtday. At what rate was the money invested, if simple interest was calculated ?
4. Bongani buys a dining room table costing R 8 500 on Hire Purchase. He is charged simple interest at 17,5% per annum over 3 years.
   (a) How much will Bongani pay in total ?
   (b) How much interest does he pay ?
   (c) What is his montly installment ?

8.5 Compound Interest

To explain the concept of compound interest, the following example is discussed:

Worked Example 13: Using Simple Interest to lead to the concept Compound Interest

Question: If I deposit R1 000 into a special bank account which pays a Simple Interest of 7%. What if I empty the bank account after a year, and then take the principal and the interest and invest it back into the same account again. Then I take it all out at the end of the second year, and then put it all back in again? And then I take it all out at the end of 3 years?

Answer

Step 1 : Determine what is given and what is required

- opening balance, $P = \text{R}1\ 000$
- interest rate, $i = 7\%$

- period of time, 1 year at a time, for 3 years

We are required to find the closing balance at the end of three years.

Step 2 : Determine how to approach the problem

We know that:

$$\text{Closing Balance} = P(1 + i \cdot n)$$

Step 3 : Determine the closing balance at the end of the first year

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R1 000}(1 + 1 \times 7\%)\\ &= \text{R1 070}\end{aligned}$$

Step 4 : Determine the closing balance at the end of the second year

After the first year, we withdraw all the money and re-deposit it. The opening balance for the second year is therefore R1 070, because this is the balance after the first year.

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R1 070}(1 + 1 \times 7\%)\\ &= \text{R1 144,90}\end{aligned}$$

Step 5 : Determine the closing balance at the end of the third year

After the second year, we withdraw all the money and re-deposit it. The opening balance for the third year is therefore R1 144,90, because this is the balance after the first year.

$$\begin{aligned}\text{Closing Balance} &= P(1 + i \cdot n)\\ &= \text{R1 144,90}(1 + 1 \times 7\%)\\ &= \text{R1 225,04}\end{aligned}$$

Step 6 : Write the final answer

The closing balance after withdrawing the all the money and re-depositing each year for 3 years of saving R1 000 at an interest rate of 7% is R1 225,04.

In the two worked examples using simple interest, we have basically the same problem because P=R1 000, i=7% and n=3 years for both problems. Except in the second situation, we end up with R1 225,04 which is more than R1 210 from the first example. What has changed?

In the first example I earned R70 interest each year - the same in the first, second and third year. But in the second situation, when I took the money out and then re-invested it, I was actually earning interest in the second year on my interest (R70) from the first year. (And interest on the interest on my interest in the third year!)

This more realistically reflects what happens in the real world, and is known as Compound Interest. It is this concept which underlies just about everything we do - so we will look at more closely next.

Definition: Compound Interest

Compound interest is the interest payable on the principal and its accumulated interest.

Compound interest is a double edged sword, though - great if you are earning interest on cash you have invested, but crippling if you are stuck having to pay interest on money you have borrowed!

In the same way that we developed a formula for Simple Interest, let us find one for Compound Interest.

If our opening balance is P and we have an interest rate of i then, the closing balance at the end of the first year is:

$$\text{Closing Balance after 1 year} = P(1+i)$$

This is the same as Simple Interest because it only covers a single year. Then, if we take that out and re-invest it for another year - just as you saw us doing in the worked example above - then the balance after the second year will be:

$$\begin{aligned}\text{Closing Balance after 2 years} &= [P(1+i)] \times (1+i) \\ &= P(1+i)^2\end{aligned}$$

And if we take that money out, then invest it for another year, the balance becomes:

$$\begin{aligned}\text{Closing Balance after 3 years} &= [P(1+i)^2] \times (1+i) \\ &= P(1+i)^3\end{aligned}$$

We can see that the power of the term $(1+i)$ is the same as the number of years. Therefore,

$$\text{Closing Balance after } n \text{ years} = P(1+i)^n \tag{8.3}$$

8.5.1 Fractions add up to the Whole

It is easy to show that this formula works even when n is a fraction of a year. For example, let us invest the money for 1 month, then for 4 months, then for 7 months.

$$\begin{aligned}\text{Closing Balance after 1 month} &= P(1+i)^{\frac{1}{12}} \\ \text{Closing Balance after 5 months} &= \text{Closing Balance after 1 month invested for 4 months more} \\ &= [P(1+i)^{\frac{1}{12}}]^{\frac{4}{12}} \\ &= P(1+i)^{\frac{1}{12}+\frac{4}{12}} \\ &= P(1+i)^{\frac{5}{12}} \\ \text{Closing Balance after 12 months} &= \text{Closing Balance after 5 month invested for 7 months more} \\ &= [P(1+i)^{\frac{5}{12}}]^{\frac{7}{12}} \\ &= P(1+i)^{\frac{5}{12}+\frac{7}{12}} \\ &= P(1+i)^{\frac{12}{12}} \\ &= P(1+i)^1\end{aligned}$$

which is the same as investing the money for a year.

Look carefully at the long equation above. It is not as complicated as it looks! All we are doing is taking the opening amount (P), then adding interest for just 1 month. Then we are taking that new balance and adding interest for a further 4 months, and then finally we are taking the new balance after a total of 5 months, and adding interest for 7 more months. Take a look again, and check how easy it really is.

Does the final formula look familiar? Correct - it is the same result as you would get for simply investing P for one full year. This is exactly what we would expect, because:

$$1 \text{ month} + 4 \text{ months} + 7 \text{ months} = 12 \text{ months},$$

which is a year. Can you see that? Do not move on until you have understood this point.

8.5.2 The Power of Compound Interest

To see how important this "interest on interest" is, we shall compare the difference in closing balances for money earning simple interest and money earning compound interest. Consider an amount of R10 000 that you have to invest for 10 years, and assume we can earn interest of 9%. How much would that be worth after 10 years?

The closing balance for the money earning simple interest is:

$$\begin{aligned}\text{Closing Balance} &= P(1+i\cdot n)\\ &= \text{R}10\ 000(1+9\%\times 10)\\ &= \text{R}19\ 000\end{aligned}$$

The closing balance for the money earning compound interest is:

$$\begin{aligned}\text{Closing Balance} &= P(1+i)^n)\\ &= \text{R}10\ 000(1+9\%)^{10}\\ &= \text{R}23\ 673{,}64\end{aligned}$$

So next time someone talks about the "magic of compound interest", not only will you know what they mean - but you will be able to prove it mathematically yourself!

Again, keep in mind that this is good news and bad news. When you are earning interest on money you have invested, compound interest helps that amount to increase exponentially. But if you have borrowed money, the build up of the amount you owe will grow exponentially too.

Worked Example 14: Taking out a Loan

Question: Mr Lowe wants to take out a loan of R 350 000. He does not want to pay back more than R625 000 altogether on the loan. If the interest rate he is offered is 13%, over what period should he take the loan

Answer

Step 1 : Determine what has been provided and what is required

- opening balance, $P = \text{R}350\ 000$
- closing balance, $A = \text{R}625\ 000$
- interest rate, $i = 13\%\ \text{peryear}$

We are required to find the time period(n).

Step 2 : Determine how to approach the problem

We know from (8.3) that:

$$\text{Closing Balance,(A)} = P(1+i)^n$$

We need to find n.

Therefore we covert the formula to:

$$\frac{\text{A}}{\text{P}} = (1+i)^n$$

and then find n by trial and error.

Step 3 : Solve the problem

$$\begin{aligned}\frac{\text{A}}{\text{P}} &= (1+i)^n\\ \frac{625000}{350000} &= (1+0{,}13)^n\\ 1{,}785... &= (1{,}13)^n\end{aligned}$$

$$\begin{aligned}\text{Try n} &= 3: \quad (1{,}13)^3 = 1{,}44...\\ \text{Try n} &= 4: \quad (1{,}13)^4 = 1{,}63...\\ \text{Try n} &= 5: \quad (1{,}13)^5 = 1{,}84...\end{aligned}$$

Step 4 : Write the final answer

Mr Lowe should take the loan over four years

8.5.3 Other Applications of Compound Growth

Worked Example 15: Population Growth

Question: South Africa's population is increasing by 2,5% per year. If the current population is 43 million, how many more people will there be in South Africa in two year's time ?

Answer

Step 1 : Determine what has been provided and what is required

- opening balance, $P = 43\ 000\ 000$
- period of time, $n = 2$ year
- interest rate, $i = 2{,}5\%$ peryear

We are required to find the closing balance(A).

Step 2 : Determine how to approach the problem

We know from (8.3) that:

$$\text{Closing Balance,(A)} = P(1+i)^n$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1+i)^n \\ &= 43\ 000\ 000(1+0{,}025)^2 \\ &= 45\ 176\ 875 \end{aligned}$$

Step 4 : Write the final answer

There are $45\ 176\ 875 - 43\ 000\ 000 = 2\ 176\ 875$ more people in 2 year's time

Worked Example 16: Compound Decrease

Question: A swimming pool is being treated for a build-up of algae. Initially, $50m^2$ of the pool is covered by algae. With each day of treatment, the algae reduces by 5%. What area is covered by algae after 30 days of treatment ?

Answer

Step 1 : Determine what has been provided and what is required

- opening balance, $P = 50\text{m}^2$
- period of time, $n = 30$ days
- interest rate, $i = 5\%$ perday

We are required to find the closing balance(A).

Step 2 : Determine how to approach the problem

We know from (8.3) that:

$$\text{Closing Balance,(A)} = P(1+i)^n$$

But this is compound **decrease** so we can use the formula:

$$\text{Closing Balance,(A)} = P(1-i)^n$$

Step 3 : Solve the problem

$$\begin{aligned} \text{A} &= P(1-i)^n \\ &= 50(1-0{,}05)^{30} \\ &= 10{,}73m^2 \end{aligned}$$

Step 4 : Write the final answer
Therefore the area still covered with algae is $10{,}73m^2$

?

Exercise: Compound Interest

1. An amount of R3 500 is invested in a savings account which pays compound interest at a rate of 7,5% per annum. Calculate the balance accumulated by the end of 2 years.
2. If the average rate of inflation for the past few years was 7,3% and your water and electricity account is R 1 425 on average, what would you expect to pay in 6 years time ?
3. Shrek wants to invest some money at 11% per annum compound interest. How much money (to the nearest rand) should he invest if he wants to reach a sum of R 100 000 in five year's time ?

8.6 Summary

As an easy reference, here are the key formulae that we derived and used during this chapter. While memorising them is nice (there are not many), it is the application that is useful. Financial experts are not paid a salary in order to recite formulae, they are paid a salary to use the right methods to solve financial problems.

8.6.1 Definitions

P	Principal (the amount of money at the starting point of the calculation)
i	interest rate, normally the effective rate per annum
n	period for which the investment is made

8.6.2 Equations

$$\left.\begin{array}{l} \text{Closing Balance - simple interest} \\ \text{Solve for } i \\ \text{Solve for } n \end{array}\right\} = P(1 + i \cdot n)$$

$$\left.\begin{array}{l}\text{Closing Balance - compound interest}\\ \text{Solve for } i\\ \text{Solve for } n\end{array}\right\} = P(1+i)^n$$

Important: Always keep the interest and the time period in the same units of time (e.g. both in years, or both in months etc.).

8.7 End of Chapter Exercises

1. You are going on holiday to Europe. Your hotel will cost €200 per night. How much will you need in Rands to cover your hotel bill, if the exchange rate is €1 = R9,20.
2. Calculate how much you will earn if you invested R500 for 1 year at the following interest rates:
   (a) 6,85% simple interest
   (b) 4,00% compound interest
3. Bianca has R1 450 to invest for 3 years. Bank A offers a savings account which pays simple interest at a rate of 11% per annum, whereas Bank B offers a savings account paying compound interest at a rate of 10,5% per annum. Which account would leave Bianca with the highest accumulated balance at the end of the 3 year period?
4. How much simple interest is payable on a loan of R2 000 for a year, if the interest rate is 10%?
5. How much compound interest is payable on a loan of R2 000 for a year, if the interest rate is 10%?
6. Discuss:
   (a) Which type of interest would you like to use if you are the borrower?
   (b) Which type of interest would you like to use if you were the banker?
7. Calculate the compound interest for the following problems.
   (a) A R2 000 loan for 2 years at 5%.
   (b) A R1 500 investment for 3 years at 6%.
   (c) An R800 loan for l year at 16%.
8. If the exchange rate 100 Yen = R 6,2287 and 1 AUD = R 5,1094 , determine the exchange rate between the Australian Dollar and the Japanese Yen.
9. Bonnie bought a stove for R 3 750. After 3 years she paid for it and the R 956,25 interest that was charged for hire-purchase. Determine the simple rate of interest that was charged.

Chapter 9

Products and Factors - Grade 10

9.1 Introduction

In this chapter you will learn how to work with algebraic expressions. You will recap some of the work on factorisation and multiplying out expressions that you learnt in earlier grades. This work will then be extended upon for Grade 10.

9.2 Recap of Earlier Work

The following should be familiar. Examples are given as reminders.

9.2.1 Parts of an Expression

Mathematical expressions are just like sentences and their parts have special names. You should be familiar with the following names used to describe the parts of an mathematical expression.

$$a \cdot x^k + b \cdot x + c^m = 0 \quad (9.1)$$

$$d \cdot y^p + e \cdot y + f \leq 0 \quad (9.2)$$

Name	Examples (separated by commas)
term	$a \cdot x^k$,$b \cdot x$, c^m, $d \cdot y^p$, $e \cdot y$, f
expression	$a \cdot x^k + b \cdot x + c^m$, $d \cdot y^p + e \cdot y + f$
coefficient	a, b, d, e
exponent (or index)	k, p
base	x, y, c
constant	a, b, c, d, e, f
variable	x, y
equation	$a \cdot x^k + b \cdot x + c^m = 0$
inequality	$d \cdot y^p + e \cdot y + f \leq 0$
binomial	expression with two terms
trinomial	expression with three terms

9.2.2 Product of Two Binomials

A *binomial* is a mathematical expression with two terms, e.g. $(ax + b)$ and $(cx + d)$. If these two binomials are multiplied, the following is the result:

$$\begin{aligned}(a \cdot x + b)(c \cdot x + d) &= (ax)(c \cdot x + d) + b(c \cdot x + d) \\ &= (ax)(cx) + (ax)d + b(cx) + b \cdot d\end{aligned}$$

Worked Example 17: Product of two Binomials

Question: Find the product of $(3x - 2)(5x + 8)$
Answer

$$\begin{aligned}(3x - 2)(5x + 8) &= (3x)(5x) + (3x)(8) + (-2)(5x) + (-2)(8) \\ &= 15x^2 + 24x - 10x - 16 \\ &= 15x^2 + 14x - 16\end{aligned}$$

The product of two identical binomials is known as the *square of the binomials* and is written as:

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

If the two terms are $ax + b$ and $ax - b$ then their product is:

$$(ax + b)(ax - b) = a^2x^2 - b^2$$

This is known as the *difference of squares*.

9.2.3 Factorisation

Factorisation is the opposite of expanding brackets. For example expanding brackets would require $2(x+1)$ to be written as $2x+2$. Factorisation would be to start with $2x+2$ and to end up with $2(x+1)$. In previous grades you factorised based on common factors and on difference of squares.

Common Factors

Factorising based on common factors relies on there being common factors between your terms. For example, $2x - 6x^2$ can be factorised as follows:

$$2x - 6x^2 = 2x(1 - 3x)$$

Activity :: Investigation : Common Factors
Find the highest common factors of the following pairs of terms:

(a) $6y; 18x$	(b) $12mn; 8n$	(c) $3st; 4su$	(d) $18kl; 9kp$	(e) $abc; ac$
(f) $2xy; 4xyz$	(g) $3uv; 6u$	(h) $9xy; 15xz$	(i) $24xyz; 16yz$	(j) $3m; 45n$

Difference of Squares

We have seen that:

$$(ax+b)(ax-b) = a^2x^2 - b^2 \tag{9.3}$$

Since 9.3 is an equation, both sides are always equal. This means that an expression of the form:

$$a^2x^2 - b^2$$

can be factorised to

$$(ax+b)(ax-b)$$

Therefore,

$$a^2x^2 - b^2 = (ax+b)(ax-b)$$

For example, $x^2 - 16$ can be written as $(x^2 - 4^2)$ which is a difference of squares. Therefore the factors of $x^2 - 16$ are $(x-4)$ and $(x+4)$.

Worked Example 18: Factorisation

Question: Factorise completely: $b^2y^5 - 3aby^3$
Answer

$$b^2y^5 - 3aby^3 = by^3(by^2 - 3a)$$

Worked Example 19: Factorising binomials with a common bracket

Question: Factorise completely: $3a(a-4) - 7(a-4)$
Answer
Step 1 : bracket $(a-4)$ is the common factor

$$3a(a-4) - 7(a-4) = (a-4)(3a-7)$$

Worked Example 20: Factorising using a switch around in brackets

Question: Factorise $5(a-2) - b(2-a)$
Answer
Step 1 : Note that $(2-a) = -(a-2)$

$$\begin{aligned} 5(a-2) - b(2-a) &= 5(a-2) - [-b(a-2)] \\ &= 5(a-2) + b(a-2) \\ &= (a-2)(5+b) \end{aligned}$$

Exercise: Recap

1. Find the products of:

(a) $2y(y+4)$	(b) $(y+5)(y+2)$	(c) $(y+2)(2y+1)$
(d) $(y+8)(y+4)$	(e) $(2y+9)(3y+1)$	(f) $(3y-2)(y+6)$

2. Factorise:
   (a) $2l+2w$
   (b) $12x+32y$
   (c) $6x^2+2x+10x^3$
   (d) $2xy^2+xy^2z+3xy$
   (e) $-2ab^2-4a^2b$
3. Factorise completely:

(a) $7a+4$	(b) $20a-10$	(c) $18ab-3bc$
(d) $12kj+18kq$	(e) $16k^2-4k$	(f) $3a^2+6a-18$
(g) $-6a-24$	(h) $-2ab-8a$	(i) $24kj-16k^2j$
(j) $-a^2b-b^2a$	(k) $12k^2j+24k^2j^2$	(l) $72b^2q-18b^3q^2$
(m) $4(y-3)+k(3-y)$	(n) $a(a-1)-5(a-1)$	(o) $bm(b+4)-6m(b+4)$
(p) $a^2(a+7)+a(a+7)$	(q) $3b(b-4)-7(4-b)$	(r) $a^2b^2c^2-1$

9.3 More Products

We have seen how to multiply two binomials in section 9.2.2. In this section we learn how to multiply a binomial (expression with two terms) by a trinomial (expression with three terms). Fortunately, we use the same methods we used to multiply two binomials to multiply a binomial and a trinomial.

For example, multiply $2x+1$ by x^2+2x+1.

$$\begin{aligned}
& (2x+1)(x^2+2x+1) \\
=\ & 2x(x^2+2x+1)+1(x^2+2x+1) \quad \text{(apply distributive law)} \\
=\ & [2x(x^2)+2x(2x)+2x(1)]+[1(x^2)+1(2x)+1(1)] \\
=\ & 4x^3+4x^2+2x+x^2+2x+1 \quad \text{(expand the brackets)} \\
=\ & 4x^3+(4x^2+x^2)+(2x+2x)+1 \quad \text{(group like terms to simplify)} \\
=\ & 4x^3+5x^2+4x+1 \quad \text{(simplify to get final answer)}
\end{aligned}$$

Important: Multiplication of Binomial with Trinomial

If the binomial is $A+B$ and the trinomial is $C+D+E$, then the very first step is to apply the distributive law:

$$(A+B)(C+D+E)=A(C+D+E)+B(C+D+E) \tag{9.4}$$

If you remember this, you will never go wrong!

Worked Example 21: Multiplication of Binomial with Trinomial

Question: Multiply $x - 1$ with $x^2 - 2x + 1$.
Answer
Step 1 : Determine what is given and what is required
We are given two expressions: a binomial, $x - 1$, and a trinomial, $x^2 - 2x + 1$. We need to multiply them together.
Step 2 : Determine how to approach the problem
Apply the distributive law and then simplify the resulting expression.
Step 3 : Solve the problem

$$\begin{aligned}
& (x-1)(x^2-2x+1) \\
= \;& x(x^2-2x+1) - 1(x^2-2x+1) \quad \text{(apply distributive law)} \\
= \;& [x(x^2) + x(-2x) + x(1)] + [-1(x^2) - 1(-2x) - 1(1)] \\
= \;& x^3 - 2x^2 + x - x^2 + 2x - 1 \quad \text{(expand the brackets)} \\
= \;& x^3 + (-2x^2 - x^2) + (x + 2x) - 1 \quad \text{(group like terms to simplify)} \\
= \;& x^3 - 3x^2 + 3x - 1 \quad \text{(simplify to get final answer)}
\end{aligned}$$

Step 4 : Write the final answer
The product of $x - 1$ and $x^2 - 2x + 1$ is $x^3 - 3x^2 + 3x - 1$.

Worked Example 22: Sum of Cubes

Question: Find the product of $x + y$ and $x^2 - xy + y^2$.
Answer
Step 1 : Determine what is given and what is required
We are given two expressions: a binomial, $x + y$, and a trinomial, $x^2 - xy + y^2$. We need to multiply them together.
Step 2 : Determine how to approach the problem
Apply the distributive law and then simplify the resulting expression.
Step 3 : Solve the problem

$$\begin{aligned}
& (x+y)(x^2-xy+y^2) \\
= \;& x(x^2-xy+y^2) + y(x^2-xy+y^2) \quad \text{(apply distributive law)} \\
= \;& [x(x^2) + x(-xy) + x(y^2)] + [y(x^2) + y(-xy) + y(y^2)] \\
= \;& x^3 - x^2y + xy^2 + yx^2 - xy^2 + y^3 \quad \text{(expand the brackets)} \\
= \;& x^3 + (-x^2y + yx^2) + (xy^2 - xy^2) + y^3 \quad \text{(group like terms to simplify)} \\
= \;& x^3 + y^3 \quad \text{(simplify to get final answer)}
\end{aligned}$$

Step 4 : Write the final answer
The product of $x + y$ and $x^2 - xy + y^2$ is $x^3 + y^3$.

Important: We have seen that:

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3$$

This is known as a *sum of cubes*.

Activity :: Investigation : Difference of Cubes
Show that the difference of cubes ($x^3 - y^3$) is given by the product of $x - y$ and $x^2 + xy + y^2$.

Exercise: Products

1. Find the products of:

(a) $(-2y^2 - 4y + 11)(5y - 12)$
(b) $(-11y + 3)(-10y^2 - 7y - 9)$
(c) $(4y^2 + 12y + 10)(-9y^2 + 8y + 2)$
(d) $(7y^2 - 6y - 8)(-2y + 2)$
(e) $(10y^5 + 3)(-2y^2 - 11y + 2)$
(f) $(-12y - 3)(12y^2 - 11y + 3)$
(g) $(-10)(2y^2 + 8y + 3)$
(h) $(2y^6 + 3y^5)(-5y - 12)$
(i) $(6y^7 - 8y^2 + 7)(-4y - 3)(-6y^2 - 7y - 11)$
(j) $(-9y^2 + 11y + 2)(8y^2 + 6y - 7)$
(k) $(8y^5 + 3y^4 + 2y^3)(5y + 10)(12y^2 + 6y + 6)$
(l) $(-7y + 11)(-12y + 3)$
(m) $(4y^3 + 5y^2 - 12y)(-12y - 2)(7y^2 - 9y + 12)$
(n) $(7y + 3)(7y^2 + 3y + 10)$
(o) $(9)(8y^2 - 2y + 3)$
(p) $(-12y + 12)(4y^2 - 11y + 11)$
(q) $(-6y^4 + 11y^2 + 3y)(10y + 4)(4y - 4)$
(r) $(-3y^6 - 6y^3)(11y - 6)(10y - 10)$
(s) $(-11y^5 + 11y^4 + 11)(9y^3 - 7y^2 - 4y + 6)$
(t) $(-3y + 8)(-4y^3 + 8y^2 - 2y + 12)$

2. Remove the brackets and simplify:$(2h + 3)(4h^2 - 6h + 9)$

9.4 Factorising a Quadratic

Finding the factors of a quadratic is quite easy, and some are easier than others.

The simplest quadratic has the form ax^2, which factorises to $(x)(ax)$. For example, $25x^2$ factorises to $(5x)(5x)$ and $2x^2$ factorises to $(2x)(x)$.

The second simplest quadratic is of the form $ax^2 + bx$. We can see here that x is a common factor of both terms. Therefore, $ax^2 + bx$ factorises to $x(ax + b)$. For example, $8y^2 + 4y$ factorises to $4y(2y + 1)$.

The third simplest quadratic is made up of the difference of squares. We know that:

$$(a + b)(a - b) = a^2 - b^2.$$

This is true for any values of a and b, and more importantly since it is an equality, we can also write:

$$a^2 - b^2 = (a + b)(a - b).$$

This means that if we ever come across a quadratic that is made up of a difference of squares, we can immediately write down what the factors are.

Worked Example 23: Difference of Squares

Question: Find the factors of $9x^2 - 25$.
Answer
Step 1 : Examine the quadratic
We see that the quadratic is a difference of squares because:

$$(3x)^2 = 9x^2$$

and

$$5^2 = 25.$$

Step 2 : Write the quadratic as the difference of squares

$$9x^2 - 25 = (3x)^2 - 5^2$$

Step 3 : Write the factors

$$(3x)^2 - 5^2 = (3x - 5)(3x + 5)$$

Step 4 : Write the final answer
The factors of $9x^2 - 25$ are $(3x - 5)(3x + 5)$.

The three types of quadratic that we have seen are very simple to factorise. However, many quadratics do not fall into these categories, and we need a more general method to factorise quadratics like $x^2 - x - 2$?

We can learn about how to factorise quadratics by looking at how two binomials are multiplied to get a quadratic. For example, $(x + 2)(x + 3)$ is multiplied out as:

$$\begin{aligned}(x+2)(x+3) &= x(x+3) + 2(x+3) \\ &= (x)(x) + 3x + 2x + (2)(3) \\ &= x^2 + 5x + 6.\end{aligned}$$

We see that the x^2 term in the quadratic is the product of the x-terms in each bracket. Similarly, the 6 in the quadratic is the product of the 2 and 3 the brackets. Finally, the middle term is the sum of two terms.

So, how do we use this information to factorise the quadratic?

Let us start with factorising $x^2 + 5x + 6$ and see if we can decide upon some general rules. Firstly, write down two brackets with an x in each bracket and space for the remaining terms.

$$(\quad x \quad\quad)(\quad x \quad\quad)$$

Next decide upon the factors of 6. Since the 6 is positive, these are:

Factors of 6	
1	6
2	3
-1	-6
-2	-3

Therefore, we have four possibilities:

Option 1	Option 2	Option 3	Option 4
$(x + 1)(x + 6)$	$(x - 1)(x - 6)$	$(x + 2)(x + 3)$	$(x - 2)(x - 3)$

Next we expand each set of brackets to see which option gives us the correct middle term.

Option 1	Option 2	Option 3	Option 4
$(x+1)(x+6)$	$(x-1)(x-6)$	$(x+2)(x+3)$	$(x-2)(x-3)$
x^2+7x+6	x^2-7x+6	$\underline{x^2+5x+6}$	x^2-5x+6

We see that Option 3 (x+2)(x+3) is the correct solution. As you have seen that the process of factorising a quadratic is mostly trial and error, however the is some information that can be used to simplify the process.

Method: Factorising a Quadratic

1. First divide the entire equation by any common factor of the coefficients, so as to obtain an equation of the form $ax^2 + bx + c = 0$ where a, b and c have no common factors and a is positive.
2. Write down two brackets with an x in each bracket and space for the remaining terms.
$$(\quad x \qquad)(\quad x \qquad) \tag{9.5}$$
3. Write down a set of factors for a and c.
4. Write down a set of options for the possible factors for the quadratic using the factors of a and c.
5. Expand all options to see which one gives you the correct answer.

There are some tips that you can keep in mind:

- If c is positive, then the factors of c must be either both positive or both negative. The factors are both negative if b is negative, and are both positive if b is positive. If c is negative, it means only one of the factors of c is negative, the other one being positive.
- Once you get an answer, multiply out your brackets again just to make sure it really works.

Worked Example 24: Factorising a Quadratic

Question: Find the factors of $3x^2 + 2x - 1$.

Answer

Step 1 : Check whether the quadratic is in the form $ax^2 + bx + c = 0$ with a positive.

The quadratic is in the required form.

Step 2 : Write down two brackets with an x in each bracket and space for the remaining terms.

$$(\quad x \qquad)(\quad x \qquad) \tag{9.6}$$

Write down a set of factors for a and c. The possible factors for a are: (1,3). The possible factors for c are: (-1,1) or (1,-1).

Write down a set of options for the possible factors for the quadratic using the factors of a and c. Therefore, there are two possible options.

Option 1	Option 2
$(x-1)(3x+1)$	$(x+1)(3x-1)$
$3x^2-2x-1$	$\underline{3x^2+2x-1}$

Step 3 : Check your answer

$$\begin{aligned}(x+1)(3x-1) &= x(3x-1)+1(3x-1)\\ &= (x)(3x)+(x)(-1)+(1)(3x)+(1)(-1)\\ &= 3x^2-x+3x-1\\ &= x^2+2x-1.\end{aligned}$$

Step 4 : Write the final answer
The factors of $3x^2+2x-1$ are $(x+1)$ and $(3x-1)$.

?

Exercise: Factorising a Trinomial

1. Factorise the following:

(a) $x^2+8x+15$	(b) $x^2+10x+24$	(c) x^2+9x+8
(d) $x^2+9x+14$	(e) $x^2+15x+36$	(f) $x^2+13x+36$

2. Factorise the following:
   (a) $x^2-2x-15$
   (b) x^2+2x-3
   (c) x^2+2x-8
   (d) x^2+x-20
   (e) x^2-x-20
3. Find the factors for the following quadratic expressions:
   (a) $2x^2+11x+5$
   (b) $3x^2+19x+6$
   (c) $6x^2+7x+2$
   (d) $12x^2+7x+1$
   (e) $8x^2+6x+1$
4. Find the factors for the following trinomials:
   (a) $3x^2+17x-6$
   (b) $7x^2-6x-1$
   (c) $8x^2-6x+1$
   (d) $2x^2-5x-3$

9.5 Factorisation by Grouping

One other method of factorisation involves the use of common factors. We know that the factors of $3x+3$ are 3 and $(x+1)$. Similarly, the factors of $2x^2+2x$ are $2x$ and $(x+1)$. Therefore, if we have an expression:

$$2x^2+2x+3x+3$$

then we can factorise as:

$$2x(x+1)+3(x+1).$$

You can see that there is another common factor: $x+1$. Therefore, we can now write:

$$(x+1)(2x+3).$$

We get this by taking out the $x+1$ and see what is left over. We have a $+2x$ from the first term and a $+3$ from the second term. This is called *factorisation by grouping*.

Worked Example 25: Factorisation by Grouping

Question: Find the factors of $7x + 14y + bx + 2by$ by grouping
Answer
Step 1 : Determine if there are common factors to all terms
There are no factors that are common to all terms.
Step 2 : Determine if there are factors in common between some terms
7 is a common factor of the first two terms and b is a common factor of the second two terms.
Step 3 : Re-write expression taking the factors into account

$$7x + 14y + bx + 2by = 7(x + 2y) + b(x + 2y)$$

Step 4 : Determine if there are more common factors
$x + 2y$ is a common factor.
Step 5 : Re-write expression taking the factors into account

$$7(x + 2y) + b(x + 2y) = (x + 2y)(7 + b)$$

Step 6 : Write the final answer
The factors of $7x + 14y + bx + 2by$ are $(7 + b)$ and $(x + 2y)$.

Exercise: Factorisation by Grouping

1. Factorise by grouping: $6x + 9 + 2ax + 3$
2. Factorise by grouping: $x^2 - 6x + 5x - 30$
3. Factorise by grouping: $5x + 10y - ax - 2ay$
4. Factorise by grouping: $a^2 - 2a - ax + 2x$
5. Factorise by grouping: $5xy - 3y + 10x - 6$

9.6 Simplification of Fractions

In some cases of simplifying an algebraic expression, the expression will be a fraction. For example,

$$\frac{x^2 + 3x}{x + 3}$$

has a quadratic in the numerator and a binomial in the denominator. You can apply the different factorisation methods to simplify the expression.

$$\begin{aligned} & \frac{x^2 + 3x}{x + 3} \\ = & \frac{x(x + 3)}{x + 3} \\ = & x \quad \text{provided } x \neq -3 \end{aligned}$$

Worked Example 26: Simplification of Fractions

Question: Simplify: $\frac{2x-b+x-ab}{ax^2-abx}$

Answer

Step 1 : Factorise numerator and denominator

Use *grouping* for numerator and *common factor* for denominator in this example.

$$\begin{aligned} &= \frac{(ax-ab)+(x-b)}{ax^2-abx} \\ &= \frac{a(x-b)+(x-b)}{ax(x-b)} \\ &= \frac{(x-b)(a+1)}{ax(x-b)} \end{aligned}$$

Step 2 : Cancel out same factors

The simplified answer is:

$$= \frac{a+1}{ax}$$

Worked Example 27: Simplification of Fractions

Question: Simplify: $\frac{x^2-x-2}{x^2-4} \div \frac{x^2+x}{x^2+2x}$

Answer

Step 1 : Factorise numerators and denominators

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \div \frac{x(x+1)}{x(x+2)}$$

Step 2 : Multiply by factorised reciprocal

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \times \frac{x(x+2)}{x(x+1)}$$

Step 3 : Cancel out same factors

The simplified answer is

$$= 1$$

Exercise: Simplification of Fractions

1. Simplify:

(a) $\frac{3a}{15}$
(b) $\frac{2a+10}{4}$
(c) $\frac{5a+20}{a+4}$
(d) $\frac{a^2-4a}{a-4}$
(e) $\frac{3a^2-9a}{2a-6}$
(f) $\frac{9a+27}{9a+18}$
(g) $\frac{6ab+2a}{2b}$
(h) $\frac{16x^2y-8xy}{12x-6}$
(i) $\frac{4xyp-8xp}{12xy}$
(j) $\frac{3a+9}{14} \div \frac{7a+21}{a+3}$
(k) $\frac{a^2-5a}{2a+10} \div \frac{3a+15}{4a}$
(l) $\frac{3xp+4p}{8p} \div \frac{12p^2}{3x+4}$
(x) $\frac{16}{2xp+4x} \div \frac{6x^2+8x}{12}$
(y) $\frac{24a-8}{12} \div \frac{9a-3}{6}$
(o) $\frac{a^2+2a}{5} \div \frac{2a+4}{20}$
(p) $\frac{p^2+pq}{7p} \div \frac{8p+8q}{21q}$
(q) $\frac{5ab-15b}{4a-12} \div \frac{6b^2}{a+b}$
(r) $\frac{f^2a-fa^2}{f-a}$

2. Simplify: $\frac{x^2-1}{3} \times \frac{1}{x-1} - \frac{1}{2}$

9.7 End of Chapter Exercises

1. Factorise:

(a) $a^2 - 9$
(b) $m^2 - 36$
(c) $9b^2 - 81$
(d) $16b^6 - 25a^2$
(e) $m^2 - (1/9)$
(f) $5 - 5a^2b^6$
(g) $16ba^4 - 81b$
(h) $a^2 - 10a + 25$
(i) $16b^2 + 56b + 49$
(j) $2a^2 - 12ab + 18b^2$
(k) $-4b^2 - 144b^8 + 48b^5$
(l) $a^3 - 27$
(m) $125a^3 + b^3$
(n) $128b^7 - 250ba^6$
(o) $c^3 + 27$
(p) $64b^3 + 1$
(q) $5a^3 - 40c^3$
(r) $2b^4 - 128b$

2. Show that $(2x - 1)^2 - (x - 3)^2$ can be simplified to $(x + 2)(3x - 4)$

3. What must be added to $x^2 - x + 4$ to make it equal to $(x + 2)^2$

Chapter 10

Equations and Inequalities - Grade 10

10.1 Strategy for Solving Equations

This chapter is all about solving different types of equations for one or two variables. In general, we want to get the unknown variable alone on the left hand side of the equation with all the constants on the right hand side of the equation. For example, in the equation $x - 1 = 0$, we want to be able to write the equation as $x = 1$.

As we saw in section 2.9 (page 13), an equation is like a set of weighing scales, that must always be balanced. When we solve equations, we need to keep in mind that what is done to one side must be done to the other.

Method: Rearranging Equations

You can add, subtract, multiply or divide both sides of an equation by any number you want, as long as you always do it to both sides.

For example, in the equation $x+5-1 = -6$, we want to get x alone on the left hand side of the equation. This means we need to subtract 5 and add 1 on the left hand side. However, because we need to keep the equation balanced, we also need to subtract 5 and add 1 on the right hand side.

$$\begin{aligned} x+5-1 &= -6 \\ x+5-5-1+1 &= -6-5+1 \\ x+0+0 &= -11+1 \\ x &= -10 \end{aligned}$$

In another example, $\frac{2}{3}x = 8$, we must divide by 2 and multiply by 3 on the left hand side in order to get x alone. However, in order to keep the equation balanced, we must also divide by 2 and multiply by 3 on the right hand side.

$$\begin{aligned} \frac{2}{3}x &= 8 \\ \frac{2}{3}x \div 2 \times 3 &= 8 \div 2 \times 3 \\ \frac{2}{2} \times \frac{3}{3} \times x &= \frac{8 \times 3}{2} \\ 1 \times 1 \times x &= 12 \\ x &= 12 \end{aligned}$$

These are the basic rules to apply when simplifying any equation. In most cases, these rules have to be applied more than once, before we have the unknown variable on the left hand side of the equation.

We are now ready to solve some equations!

Important: The following must also be kept in mind:

1. Division by 0 is undefined.
2. If $\frac{x}{y} = 0$, then $x = 0$ and $y \neq 0$, because division by 0 is undefined.

Activity :: Investigation : Strategy for Solving Equations
In the following, identify what is wrong.

$$\begin{aligned} 4x - 8 &= 3(x - 2) \\ 4(x - 2) &= 3(x - 2) \\ \frac{4(x - 2)}{(x - 2)} &= \frac{3(x - 2)}{(x - 2)} \\ 4 &= 3 \end{aligned}$$

10.2 Solving Linear Equations

The simplest equation to solve is a linear equation. A linear equation is an equation where the power on the variable(letter, e.g. x) is 1(one). The following are examples of linear equations.

$$\begin{aligned} 2x + 2 &= 1 \\ \frac{2 - x}{3x + 1} &= 2 \\ \frac{4}{3}x - 6 &= 7x + 2 \end{aligned}$$

In this section, we will learn how to find the value of the variable that makes both sides of the linear equation true. For example, what value of x makes both sides of the very simple equation, $x + 1 = 1$ true.

Since the highest power on the variable is one(1) in a linear equation, there is at most *one solution* or *root* for the equation.

This section relies on all the methods we have already discussed: multiplying out expressions, grouping terms and factorisation. Make sure that you are comfortable with these methods, before trying out the work in the rest of this chapter.

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \quad \text{(like terms together)} \\ 2x &= -1 \quad \text{(simplified as much a possible)} \end{aligned}$$

Now we see that $2x = -1$. This means if we divide both sides by 2, we will get:

$$x = -\frac{1}{2}$$

If we substitute $x = -\frac{1}{2}$, back into the original equation, we get:

$$\begin{aligned} & 2x + 2 \\ = & \ 2(-\frac{1}{2}) + 2 \\ = & \ -1 + 2 \\ = & \ 1 \end{aligned}$$

That is all that there is to solving linear equations.

Important: Solving Equations

When you have found the solution to an equation, substitute the solution into the original equation, to check your answer.

Method: Solving Equations

The general steps to solve equations are:

1. Expand(Remove) all brackets.
2. "Move" all terms with the variable to the left hand side of equation, and all constant terms (the numbers) to the right hand side of the equal to-sign. Bearing in mind that the sign of the terms will chance(from (+) to (-) or vice versa, as they "cross over" the equal to sign.
3. Group all like terms together and simplify as much as possible.
4. Factorise if necessary.
5. Find the solution.
6. Substitute solution into **original** equation to check answer.

Worked Example 28: Solving Linear Equations

Question: Solve for x: $4 - x = 4$
Answer
Step 1 : Determine what is given and what is required
We are given $4 - x = 4$ and are required to solve for x.
Step 2 : Determine how to approach the problem
Since there are no brackets, we can start with grouping like terms and then simplifying.
Step 3 : Solve the problem

$$\begin{aligned} 4 - x &= 4 \\ -x &= 4 - 4 \quad \text{(move all constant terms (numbers) to the RHS (right hand side))} \\ -x &= 0 \quad \text{(group like terms together)} \\ -x &= 0 \quad \text{(simplify grouped terms)} \\ -x &= 0 \\ \therefore \quad x &= 0 \end{aligned}$$

Step 4 : Check the answer

Substitute solution into original equation:

$$4 - 0 = 4$$

$$4 = 4$$

Since both sides are equal, the answer is correct.

Step 5 : Write the Final Answer

The solution of $4 - x = 4$ is $x = 0$.

Worked Example 29: Solving Linear Equations

Question: Solve for x: $4(2x - 9) - 4x = 4 - 6x$

Answer

Step 1 : Determine what is given and what is required

We are given $4(2x - 9) - 4x = 4 - 6x$ and are required to solve for x.

Step 2 : Determine how to approach the problem

We start with expanding the brackets, then grouping like terms and then simplifying.

Step 3 : Solve the problem

$$\begin{aligned}
4(2x-9)-4x &= 4-6x \\
8x-36-4x &= 4-6x && \text{(expand the brackets)} \\
8x-4x+6x &= 4+36 && \text{(move all terms with x to the LHS and all constant terms to the RHS of the =)} \\
(8x-4x+6x) &= (4+36) && \text{(group like terms together)} \\
10x &= 40 && \text{(simplify grouped terms)} \\
\frac{10}{10}x &= \frac{40}{10} && \text{(divide both sides by 10)} \\
x &= 4
\end{aligned}$$

Step 4 : Check the answer

Substitute solution into original equation:

$$\begin{aligned}
4(2(4)-9)-4(4) &= 4-6(4) \\
4(8-9)-16 &= 4-24 \\
4(-1)-16 &= -20 \\
-4-16 &= -20 \\
-20 &= -20
\end{aligned}$$

Since both sides are equal to -20, the answer is correct.

Step 5 : Write the Final Answer

The solution of $4(2x - 9) - 4x = 4 - 6x$ is $x = 4$.

Worked Example 30: Solving Linear Equations

Question: Solve for x: $\frac{2-x}{3x+1} = 2$

Answer

Step 1 : Determine what is given and what is required
We are given $\frac{2-x}{3x+1} = 2$ and are required to solve for x.
Step 2 : Determine how to approach the problem
Since there is a denominator of (3x+1), we can start by multiplying both sides of the equation by (3x+1). But because division by 0 is not permissible, there is a restriction on a value for x. ($x \neq \frac{-1}{3}$)
Step 3 : Solve the problem

$$\begin{aligned}
\frac{2-x}{3x+1} &= 2 \\
(2-x) &= 2(3x+1) \\
2-x &= 6x+2 \quad \text{(remove/expand brackets)} \\
-x-6x &= 2-2 \quad \text{(move all terms containing x to the LHS and all constant terms (numbers) to the RHS.} \\
-7x &= 0 \qquad \text{(simplify grouped terms)} \\
x &= 0 \div (-7) \\
therefore \quad x &= 0 \qquad \text{zero divide by any number is 0}
\end{aligned}$$

Step 4 : Check the answer
Substitute solution into original equation:

$$\begin{aligned}
\frac{2-(0)}{3(0)+1} &= 2 \\
\frac{2}{1} &= 2
\end{aligned}$$

Since both sides are equal to 2, the answer is correct.

Step 5 : Write the Final Answer
The solution of $\frac{2-x}{3x+1} = 2$ is $x = 0$.

Worked Example 31: Solving Linear Equations

Question: Solve for x: $\frac{4}{3}x - 6 = 7x + 2$
Answer
Step 1 : Determine what is given and what is required
We are given $\frac{4}{3}x - 6 = 7x + 2$ and are required to solve for x.
Step 2 : Determine how to approach the problem
We start with multiplying each of the terms in the equation by 3, then grouping like terms and then simplifying.
Step 3 : Solve the problem

$$\begin{aligned}
\frac{4}{3}x - 6 &= 7x + 2 \\
4x - 18 &= 21x + 6 \quad \text{(each term is multiplied by 3} \\
4x - 21x &= 6 + 18 \quad \text{(move all terms with } x \text{ to the LHS and all constant terms to the RHS of the =)} \\
-17x &= 24 \qquad \text{(simplify grouped terms)} \\
\frac{-17}{-17}x &= \frac{24}{-17} \qquad \text{(divide both sides by -17)} \\
x &= \frac{-24}{17}
\end{aligned}$$

Step 4 : Check the answer
Substitute solution into original equation:

$$\begin{aligned}
\frac{4}{3} \times \frac{-24}{17} - 6 &= 7 \times \frac{-24}{17} + 2 \\
\frac{4 \times (-8)}{(17)} - 6 &= \frac{7 \times (-24)}{17} + 2 \\
\frac{(-32)}{17} - 6 &= \frac{-168}{17} + 2 \\
\frac{-32 - 102}{17} &= \frac{(-168) + 34}{17} \\
\frac{-134}{17} &= \frac{-134}{17}
\end{aligned}$$

Since both sides are equal to $\frac{-134}{17}$, the answer is correct.

Step 5 : Write the Final Answer
The solution of $\frac{4}{3}x - 6 = 7x + 2$ is, $x = \frac{-24}{17}$.

?

Exercise: Solving Linear Equations

1. Solve for y: $2y - 3 = 7$
2. Solve for w: $-3w = 0$
3. Solve for z: $4z = 16$
4. Solve for t: $12t + 0 = 144$
5. Solve for x: $7 + 5x = 62$
6. Solve for y: $55 = 5y + \frac{3}{4}$
7. Solve for z: $5z = 3z + 45$
8. Solve for a: $23a - 12 = 6 + 2a$
9. Solve for b: $12 - 6b + 34b = 2b - 24 - 64$
10. Solve for c: $6c + 3c = 4 - 5(2c - 3)$.
11. Solve for p: $18 - 2p = p + 9$
12. Solve for q: $\frac{4}{q} = \frac{16}{24}$
13. Solve for q: $\frac{4}{1} = \frac{q}{2}$
14. Solve for r: $-(-16 - r) = 13r - 1$
15. Solve for d: $6d - 2 + 2d = -2 + 4d + 8$
16. Solve for f: $3f - 10 = 10$
17. Solve for v: $3v + 16 = 4v - 10$
18. Solve for k: $10k + 5 + 0 = -2k + -3k + 80$
19. Solve for j: $8(j - 4) = 5(j - 4)$
20. Solve for m: $6 = 6(m + 7) + 5m$

10.3 Solving Quadratic Equations

A quadratic equation is an equation where the power on the variable is at most 2. The following are examples of quadratic equations.

$$\begin{aligned} 2x^2 + 2x &= 1 \\ \frac{2-x}{3x+1} &= 2x \\ \frac{4}{3}x - 6 &= 7x^2 + 2 \end{aligned}$$

Quadratic equations differ from linear equations by the fact that a linear equation only has one solution, while a quadratic equation has *at most* two solutions. There are some special situations when a quadratic equation only has one solution.

We solve quadratic equations by factorisation, that is writing the quadratic as a product of two expressions in brackets. For example, we know that:

$$(x+1)(2x-3) = 2x^2 - x - 3.$$

In order to solve:

$$2x^2 - x - 3 = 0$$

we need to be able to write $2x^2 - x - 3$ as $(x+1)(2x-3)$, which we already know how to do.

Activity :: Investigation : Factorising a Quadratic
Factorise the following quadratic expressions:

1. $x + x^2$
2. $x^2 + 1 + 2x$
3. $x^2 - 4x + 5$
4. $16x^2 - 9$
5. $4x^2 + 4x + 1$

Being able to factorise a quadratic means that you are one step away from solving a quadratic equation. For example, $x^2 - 3x - 2 = 0$ can be written as $(x-1)(x-2) = 0$. This means that both $x - 1 = 0$ and $x - 2 = 0$, which gives $x = 1$ and $x = 2$ as the two solutions to the quadratic equation $x^2 - 3x - 2 = 0$.

Method: Solving Quadratic Equations

1. First divide the entire equation by any common factor of the coefficients, so as to obtain an equation of the form $ax^2 + bx + c = 0$ where a, b and c have no common factors. For example, $2x^2 + 4x + 2 = 0$ can be written as $x^2 + 2x + 1 = 0$ by dividing by 2.
2. Write $ax^2 + bx + c$ in terms of its factors $(rx + s)(ux + v)$.
   This means $(rx + s)(ux + v) = 0$.
3. Once writing the equation in the form $(rx + s)(ux + v) = 0$, it then follows that the two solutions are $x = -\frac{s}{r}$ or $x = -\frac{u}{v}$.

Extension: Solutions of Quadratic Equations
There are two solutions to a quadratic equation, because any **one** of the values can solve the equation.

Worked Example 32: Solving Quadratic Equations

Question: Solve for x: $3x^2 + 2x - 1 = 0$

Answer

Step 1 : Find the factors of $3x^2 + 2x - 1$

As we have seen the factors of $3x^2 + 2x - 1$ are $(x + 1)$ and $(3x - 1)$.

Step 2 : Write the equation with the factors

$$(x + 1)(3x - 1) = 0$$

Step 3 : Determine the two solutions

We have

$$x + 1 = 0$$

or

$$3x - 1 = 0$$

Therefore, $x = -1$ or $x = \frac{1}{3}$.

Step 4 : Write the final answer

$3x^2 + 2x - 1 = 0$ for $x = -1$ or $x = \frac{1}{3}$.

Worked Example 33: Solving Quadratic Equations

Question: Solve for x: $\sqrt{x + 2} = x$

Answer

Step 1 : Square both sides of the equation

Both sides of the equation should be squared to remove the square root sign.

$$x + 2 = x^2$$

Step 2 : Write equation in the form $ax^2 + bx + c = 0$

$$\begin{aligned} x + 2 &= x^2 && \text{(subtract } x^2 \text{ to both sides)} \\ x + 2 - x^2 &= 0 && \text{(divide both sides by -1)} \\ -x - 2 + x^2 &= 0 \\ x^2 - x + 2 &= 0 \end{aligned}$$

Step 3 : Factorise the quadratic

$$x^2 - x + 2$$

The factors of $x^2 - x + 2$ are $(x - 2)(x + 1)$.

Step 4 : Write the equation with the factors

$$(x - 2)(x + 1) = 0$$

Step 5 : Determine the two solutions

We have

$$x + 1 = 0$$

or

$$x - 2 = 0$$

Therefore, $x = -1$ or $x = 2$.

Step 6 : Check whether solutions are valid
Substitute $x = -1$into the original equation $\sqrt{x+2} = x$:

$$\begin{aligned} LHS &= \sqrt{(-1)+2} \\ &= \sqrt{1} \\ &= 1 \\ but & \\ RHS &= (-1) \end{aligned}$$

Therefore LHS$\neq$RHS
Therefore $x \neq -1$
Now substitute $x = 2$ into original equation $\sqrt{x+2} = x$:

$$\begin{aligned} LHS &= \sqrt{2+2} \\ &= \sqrt{4} \\ &= 2 \\ and & \\ RHS &= 2 \end{aligned}$$

Therefore LHS = RHS
Therefore $x = 2$ is the only valid solution
Step 7 : Write the final answer
$\sqrt{x+2} = x$ for $x = 2$ only.

Worked Example 34: Solving Quadratic Equations

Question: Solve the equation: $x^2 + 3x - 4 = 0$.
Answer
Step 1 : Check if the equation is in the form $ax^2 + bx + c = 0$
The equation is in the required form, with $a = 1$.
Step 2 : Factorise the quadratic
You need the factors of 1 and 4 so that the middle term is $+3$ So the factors are:
$(x - 1)(x + 4)$
Step 3 : Solve the quadratic equation

$$x^2 + 3x - 4 = (x - 1)(x + 4) = 0 \tag{10.1}$$

Therefore $x = 1$ or $x = -4$.
Step 4 : Write the final solution
Therefore the solutions are $x = 1$ or $x = -4$.

Worked Example 35: Solving Quadratic Equations

Question: Find the roots of the quadratic equation $0 = -2x^2 + 4x - 2$.
Answer
Step 1 : Determine whether the equation is in the form $ax^2 + bx + c = 0$, with no common factors.
There is a common factor: -2. Therefore, divide both sides of the equation by -2.

$$\begin{aligned} -2x^2 + 4x - 2 &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

Step 2 : Factorise $x^2 - 2x + 1$
The middle term is negative. Therefore, the factors are $(x - 1)(x - 1)$
If we multiply out $(x - 1)(x - 1)$, we get $x^2 - 2x + 1$.
Step 3 : Solve the quadratic equation

$$x^2 - 2x + 1 = (x - 1)(x - 1) = 0$$

In this case, the quadratic is a perfect square, so there is only one solution for x: $x = 1$.
Step 4 : Write the final solution
The root of $0 = -2x^2 + 4x - 2$ is $x = 1$.

Exercise: Solving Quadratic Equations

1. Solve for x: $(3x + 2)(3x - 4) = 0$
2. Solve for a: $(5a - 9)(a + 6) = 0$
3. Solve for x: $(2x + 3)(2x - 3) = 0$
4. Solve for x: $(2x + 1)(2x - 9) = 0$
5. Solve for x: $(2x - 3)(2x - 3) = 0$
6. Solve for x: $20x + 25x^2 = 0$
7. Solve for a: $4a^2 - 17a - 77 = 0$
8. Solve for x: $2x^2 - 5x - 12 = 0$
9. Solve for b: $-75b^2 + 290b - 240 = 0$
10. Solve for y: $2y = \frac{1}{3}y^2 - 3y + 14\frac{2}{3}$
11. Solve for θ: $\theta^2 - 4\theta = -4$
12. Solve for q: $-q^2 + 4q - 6 = 4q^2 - 5q + 3$
13. Solve for t: $t^2 = 3t$
14. Solve for w: $3w^2 + 10w - 25 = 0$
15. Solve for v: $v^2 - v + 3$
16. Solve for x: $x^2 - 4x + 4 = 0$
17. Solve for t: $t^2 - 6t = 7$
18. Solve for x: $14x^2 + 5x = 6$
19. Solve for t: $2t^2 - 2t = 12$
20. Solve for y: $3y^2 + 2y - 6 = y^2 - y + 2$

10.4 Exponential Equations of the form $ka^{(x+p)} = m$

examples solved by trial and error)

Exponential equations generally have the unknown variable as the power. The following are examples of exponential equations:

$$\begin{aligned} 2^x &= 1 \\ \frac{2^{-x}}{3^{x+1}} &= 2 \\ \frac{4}{3} - 6 &= 7^x + 2 \end{aligned}$$

You should already be familiar with exponential notation. Solving exponential equations are simple, if we remember how to apply the laws of exponentials.

Activity :: Investigation : Solving Exponential Equations
Solve the following equations by completing the table:

$2^x = 2$	x						
	-3	-2	-1	0	1	2	3
2^x							

$3^x = 9$	x						
	-3	-2	-1	0	1	2	3
3^x							

$2^{x+1} = 8$	x						
	-3	-2	-1	0	1	2	3
2^{x+1}							

10.4.1 Algebraic Solution

Definition: Equality for Exponential Functions
If a is a positive number such that $a > 0$, then:

$$a^x = a^y$$

if and only if:

$$x = y$$

.

This means that if we can write all terms in an equation with the same base, we can solve the exponential equations by equating the indices. For example take the equation $3^{x+1} = 9$. This can be written as:

$$3^{x+1} = 3^2.$$

Since the bases are equal (to 3), we know that the exponents must also be equal. Therefore we can write:

$$x + 1 = 2.$$

This gives:

$$x = 1.$$

Method: Solving Exponential Equations

Try to write all terms with the same base.

Activity :: Investigation : Exponential Numbers
Write the following with the same base. The base is the first in the list. For example, in the list 2, 4, 8, the base is two and we can write 4 as 2^2.

1. 2,4,8,16,32,64,128,512,1024
2. 3,9,27,81,243
3. 5,25,125,625
4. 13,169
5. $2x$, $4x^2$, $8x^3$, $49x^8$

Worked Example 36: Solving Exponential Equations

Question: Solve for x: $2^x = 2$
Answer
Step 1 : Try to write all terms with the same base.
All terms are written with the same base.

$$2^x = 2^1$$

Step 2 : Equate the indices

$$x = 1$$

Step 3 : Check your answer

$$\begin{aligned} & 2^x \\ = & 2^{(1)} \\ = & 2^1 \end{aligned}$$

Since both sides are equal, the answer is correct.
Step 4 : Write the final answer

$$x = 1$$

is the solution to $2^x = 2$.

Worked Example 37: Solving Exponential Equations

Question: Solve:

$$2^{x+4} = 4^{2x}$$

Answer

Step 1 : Try to write all terms with the same base.

$$\begin{aligned} 2^{x+4} &= 4^{2x} \\ 2^{x+4} &= 2^{2(2x)} \\ 2^{x+4} &= 2^{4x} \end{aligned}$$

Step 2 : Equate the indices

$$x + 4 = 4x$$

Step 3 : Solve for x

$$\begin{aligned} x + 4 &= 4x \\ x - 4x &= -4 \\ -3x &= -4 \\ x &= \frac{-4}{-3} \\ x &= \frac{4}{3} \end{aligned}$$

Step 4 : Check your answer

$$\begin{aligned} \text{LHS} &= 2^{x+4} \\ &= 2^{(\frac{4}{3}+4)} \\ &= 2^{\frac{16}{3}} \\ &= (2^{16})^{\frac{1}{3}} \\ \text{RHS} &= 4^{2x} \\ &= 4^{2(\frac{4}{3})} \\ &= 4^{\frac{8}{3}} \\ &= (4^{8})^{\frac{1}{3}} \\ &= ((2^{2})^{8})^{\frac{1}{3}} \\ &= (2^{16})^{\frac{1}{3}} \\ &= \text{LHS} \end{aligned}$$

Since both sides are equal, the answer is correct.

Step 5 : Write the final answer

$$x = \frac{4}{3}$$

is the solution to $2^{x+4} = 4^{2x}$.

?

Exercise: Solving Exponential Equations

1. Solve the following exponential equations.

   a. $2^{x+5} = 2^5$
   b. $3^{2x+1} = 3^3$
   c. $5^{2x+2} = 5^3$
   d. $6^{5-x} = 6^{12}$
   e. $64^{x+1} = 16^{2x+5}$
   f. $125^x = 5$

2. Solve: $3^{9x-2} = 27$

3. Solve for k: $81^{k+2} = 27^{k+4}$
4. The growth of an algae in a pond is can be modeled by the function $f(t) = 2^t$. Find the value of t such that $f(t) = 128$?
5. Solve for x: $25^{(1-2x)} = 5^4$
6. Solve for x: $27^x \times 9^{x-2} = 1$

10.5 Linear Inequalities

graphically;

Activity :: Investigation : Inequalities on a Number Line
Represent the following on number lines:

1. $x = 4$
2. $x < 4$
3. $x \leq 4$
4. $x \geq 4$
5. $x > 4$

A linear inequality is similar to a linear equation and has the power on the variable is equal to 1. The following are examples of linear inequalities.

$$\begin{aligned} 2x + 2 &\leq 1 \\ \frac{2-x}{3x+1} &\geq 2 \\ \frac{4}{3}x - 6 &< 7x + 2 \end{aligned}$$

The methods used to solve linear inequalities are identical to those used to solve linear equations. The only difference occurs when there is a multiplication or a division that involves a minus sign. For example, we know that $8 > 6$. If both sides of the inequality are divided by -2, -4 is not greater than -3. Therefore, the inequality must switch around, making $-4 < -3$.

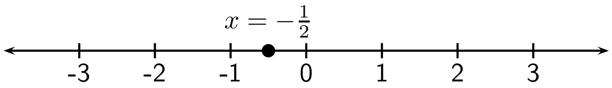
Important: When you divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes.

For example, if $x < 1$, then $-x > -1$.

In order to compare am inequality to a normal equation, we shall solve an equation first. Solve $2x + 2 = 1$.

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

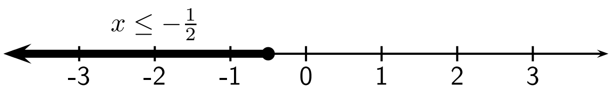
If we represent this answer on a number line, we get



Now let us solve the inequality $2x + 2 \leq 1$.

$$\begin{aligned} 2x + 2 &\leq 1 \\ 2x &\leq 1 - 2 \\ 2x &\leq -1 \\ x &\leq -\frac{1}{2} \end{aligned}$$

If we represent this answer on a number line, we get



As you can see, for the equation, there is only a single value of x for which the equation is true. However, for the inequality, there is a range of values for which the inequality is true. This is the main difference between an equation and an inequality.

Worked Example 38: Linear Inequalities

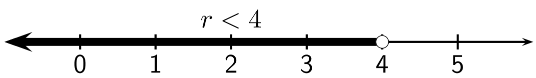
Question: Solve for r: $6 - r > 2$
Answer
Step 1 : Move all constants to the RHS

$$\begin{aligned} -r &> 2 - 6 \\ -r &> -4 \end{aligned}$$

Step 2 : Multiply both sides by -1
When you multiply by a minus sign, the direction of the inequality changes.

$$r < 4$$

Step 3 : Represent answer graphically



Worked Example 39: Linear Inequalities

Question: Solve for q: $4q + 3 < 2(q + 3)$ and represent solution on a number line.
Answer
Step 1 : Expand all brackets

$$\begin{aligned} 4q + 3 &< 2(q + 3) \\ 4q + 3 &< 2q + 6 \end{aligned}$$

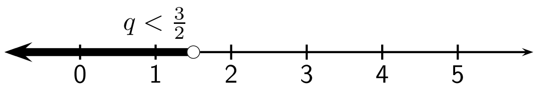
Step 2 : Move all constants to the RHS and all unknowns to the LHS

$$\begin{aligned} 4q+3 &< 2q+6 \\ 4q-2q &< 6-3 \\ 2q &< 3 \end{aligned}$$

Step 3 : Solve inequality

$$\begin{aligned} 2q &< 3 \quad \text{Divide both sides by 2} \\ q &< \frac{3}{2} \end{aligned}$$

Step 4 : Represent answer graphically

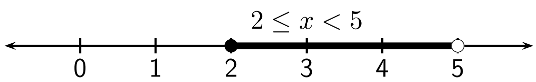


Worked Example 40: Compound Linear Inequalities

Question: Solve for x: $5 \leq x+3 < 8$ and represent solution on a number line.
Answer
Step 1 : Subtract 3 from Left, middle and right of inequalities

$$\begin{aligned} 5-3 \leq \; & x+3-3 & < 8-3 \\ 2 \leq \; & x & < 5 \end{aligned}$$

Step 2 : Represent answer graphically



Exercise: Linear Inequalities

1. Solve for x and represent the solution graphically:
   (a) $3x+4 > 5x+8$
   (b) $3(x-1)-2 \leq 6x+4$
   (c) $\frac{x-7}{3} > \frac{2x-3}{2}$
   (d) $-4(x-1) < x+2$
   (e) $\frac{1}{2}x + \frac{1}{3}(x-1) \geq \frac{5}{6}x - \frac{1}{3}$
2. Solve the following inequalities. Illustrate your answer on a number line if x is a real number.
   (a) $-2 \leq x-1 < 3$

(b) $-5 < 2x - 3 \leq 7$

3. Solve for x: $7(3x + 2) - 5(2x - 3) > 7$.
   Illustrate this answer on a number line.

10.6 Linear Simultaneous Equations

Thus far, all equations that have been encountered have one unknown variable, that must be solved for. When two unknown variables need to be solved for, two equations are required and these equations are known as simultaneous equations. The solutions to the system of simultaneous equations, are the values of the unknown variables which satisfy the system of equations simultaneously, that means at the same time. In general, if there are n unknown variables, then n equations are required to obtain a solution for each of the n variables.

An example of a system of simultaneous equations is:

$$\begin{aligned} 2x + 2y &= 1 \\ \frac{2-x}{3y+1} &= 2 \end{aligned} \qquad (10.2)$$

10.6.1 Finding solutions

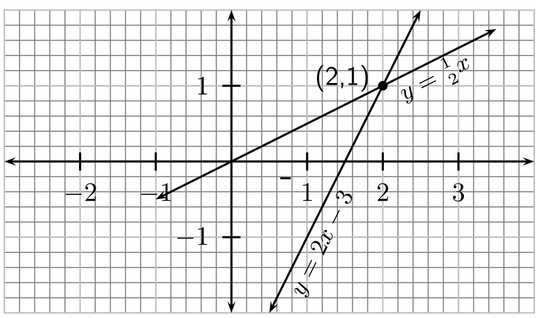
In order to find a numerical value for an unknown variable, one must have at least as many independent equations as variables. We solve simultaneous equations graphically and algebraically/

10.6.2 Graphical Solution

Simultaneous equations can also be solved graphically. If the graphs corresponding to each equation is drawn, then the solution to the system of simultaneous equations is the co-ordinate of the point at which both graphs intersect.

$$\begin{aligned} x &= 2y \\ y &= 2x - 3 \end{aligned} \qquad (10.3)$$

Draw the graphs of the two equations in (10.3).



The intersection of the two graphs is (2,1). So the solution to the system of simultaneous equations in (10.3) is $y = 1$ and $x = 2$.

This can be shown algebraically as:

$$\begin{aligned} x &= 2y \\ \therefore \quad y &= 2(2y) - 3 \\ y - 4y &= -3 \\ -3y &= -3 \\ y &= 1 \\ \text{Substitute into the first equation:} \quad x &= 2(1) \\ &= 2 \end{aligned}$$

Worked Example 41: Simultaneous Equations

Question: Solve the following system of simultaneous equations graphically.

$$\begin{aligned} 4y + 3x &= 100 \\ 4y - 19x &= 12 \end{aligned}$$

Answer

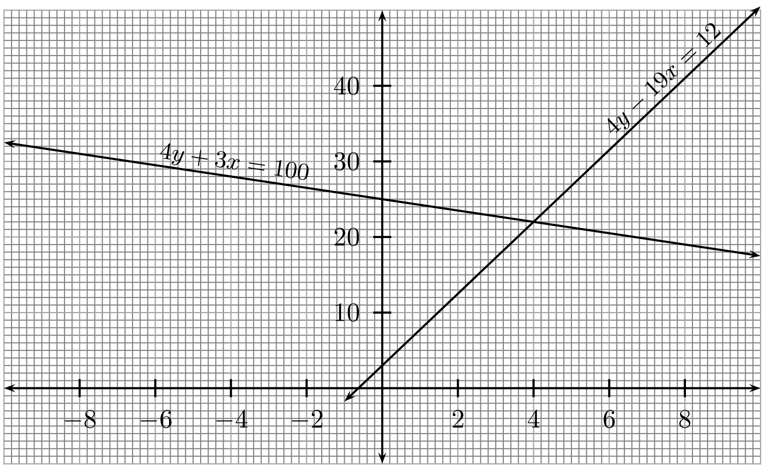
Step 1 : Draw the graphs corresponding to each equation.

For the first equation:

$$\begin{aligned} 4y + 3x &= 100 \\ 4y &= 100 - 3x \\ y &= 25 - \frac{3}{4}x \end{aligned}$$

and for the second equation:

$$\begin{aligned} 4y - 19x &= 12 \\ 4y &= 19x + 12 \\ y &= \frac{19}{4}x + 3 \end{aligned}$$



Step 2 : Find the intersection of the graphs.

The graphs intersect at (4,22).

Step 3 : Write the solution of the system of simultaneous equations as given by the intersection of the graphs.

$$\begin{aligned} x &= 4 \\ y &= 22 \end{aligned}$$

10.6.3 Solution by Substitution

A common algebraic technique is the substitution method: try to solve one of the equations for one of the variables and substitute the result into the other equations, thereby reducing the number of equations and the number of variables by 1. Continue until you reach a single equation with a single variable, which (hopefully) can be solved; back substitution then yields the values for the other variables.

In the example (**??**), we first solve the first equation for x:

$$x = \frac{1}{2} - y$$

and substitute this result into the second equation:

$$\begin{aligned}
\frac{2-x}{3y+1} &= 2 \\
\frac{2-(\frac{1}{2}-y)}{3y+1} &= 2 \\
2-(\frac{1}{2}-y) &= 2(3y+1) \\
2-\frac{1}{2}+y &= 6y+2 \\
y-6y &= -2+\frac{1}{2}+2 \\
-5y &= \frac{1}{2} \\
y &= -\frac{1}{10}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad x &= \frac{1}{2}-y \\
&= \frac{1}{2}-(-\frac{1}{10}) \\
&= \frac{6}{10} \\
&= \frac{3}{5}
\end{aligned}$$

The solution for the system of simultaneous equations (**??**) is:

$$\begin{aligned}
x &= \frac{3}{5} \\
y &= -\frac{1}{10}
\end{aligned}$$

Worked Example 42: Simultaneous Equations

Question: Solve the following system of simultaneous equations:

$$\begin{aligned}
4y+3x &= 100 \\
4y-19x &= 12
\end{aligned}$$

Answer

Step 1 : If the question, does not explicitly ask for a graphical solution, then the system of equations should be solved algebraically.

Step 2 : Make x the subject of the first equation.

$$\begin{aligned} 4y + 3x &= 100 \\ 3x &= 100 - 4y \\ x &= \frac{100 - 4y}{3} \end{aligned}$$

Step 3 : Substitute the value obtained for x into the second equation.

$$\begin{aligned} 4y - 19(\frac{100 - 4y}{3}) &= 12 \\ 12y - 19(100 - 4y) &= 36 \\ 12y - 1900 + 76y &= 36 \\ 88y &= 1936 \\ y &= 22 \end{aligned}$$

Step 4 : Substitute into the equation for x.

$$\begin{aligned} x &= \frac{100 - 4(22)}{3} \\ &= \frac{100 - 88}{3} \\ &= \frac{12}{3} \\ &= 4 \end{aligned}$$

Step 5 : Substitute the values for x and y into both equations to check the solution.

$$\begin{aligned} 4(22) + 3(4) = 88 + 12 &= 100 \quad \checkmark \\ 4(22) - 19(4) = 88 - 76 &= 12 \quad \checkmark \end{aligned}$$

Worked Example 43: Bicycles and Tricycles

Question: A shop sells bicycles and tricycles. In total there are 7 cycles and 19 wheels. Determine how many of each there are, if a bicycle has two wheels and a tricycle has three wheels.

Answer

Step 1 : Identify what is required

The number of bicycles and the number of tricycles are required.

Step 2 : Set up the necessary equations

If b is the number of bicycles and t is the number of tricycles, then:

$$\begin{aligned} b + t &= 7 \\ 2b + 3t &= 19 \end{aligned}$$

Step 3 : Solve the system of simultaneous equations using substitution.

$$\begin{aligned} b &= 7-t \\ \text{Into second equation: } 2(7-t)+3t &= 19 \\ 14-2t+3t &= 19 \\ t &= 5 \\ \text{Into first equation: } b &= 7-5 \\ &= 2 \end{aligned}$$

Step 4 : Check solution by substituting into original system of equations.

$$\begin{aligned} 2+5 &= 7 \quad \checkmark \\ 2(2)+3(5)=4+15 &= 19 \quad \checkmark \end{aligned}$$

?

Exercise: Simultaneous Equations

1. Solve graphically and confirm your answer algebraically: $3a-2b7=0$, $a-4b+1=0$
2. Solve algebraically: $15c+11d-132=0$, $2c+3d-59=0$
3. Solve algebraically: $-18e-18+3f=0$, $e-4f+47=0$
4. Solve graphically: $x+2y=7$, $x+y=0$

10.7 Mathematical Models

10.7.1 Introduction

Tom and Jane are friends. Tom picked up Jane's Physics test paper, but will not tell Jane what her marks are. He knows that Jane hates maths so he decided to tease her. Tom says: "I have 2 marks more than you do and the sum of both our marks is equal to 14. How much did we get?"

Let's help Jane find out what her marks are. We have two unknowns, Tom's mark (which we shall call t) and Jane's mark (which we shall call j). Tom has 2 more marks than Jane. Therefore,

$$t=j+2$$

Also, both marks add up to 14. Therefore,

$$t+j=14$$

The two equations make up a set of linear (because the highest power is one) simultaneous equations, which we know how to solve! Substitute for t in the second equation to get:

$$\begin{aligned} t+j &= 14 \\ j+2+j &= 14 \\ 2j+2 &= 14 \\ 2(j+1) &= 14 \\ j+1 &= 7 \\ j &= 7-1 \\ &= 6 \end{aligned}$$

Then,

$$\begin{aligned} t &= j+2 \\ &= 6+2 \\ &= 8 \end{aligned}$$

So, we see that Tom scored 8 on his test and Jane scored 6.

This problem is an example of a simple *mathematical model*. We took a problem and we able to write a set of equations that represented the problem, mathematically. The solution of the equations then gave the solution to the problem.

10.7.2 Problem Solving Strategy

The purpose of this section is to teach you the skills that you need to be able to take a problem and formulate it mathematically, in order to solve it. The general steps to follow are:

1. Read ALL of it !
2. Find out what is requested.
3. Let the requested be a variable e.g. x.
4. Rewrite the information given in terms of x. That is, translate the words into algebraic language. This is the reponse
5. Set up an equation (i.e. a mathematical sentence or model) to solve the required variable.
6. Solve the equation algebraically to find the result.

Important: Follow the three R's and solve the problem... **Request** - **Response** - **Result**

10.7.3 Application of Mathematical Modelling

Worked Example 44: Mathematical Modelling: One variable

Question: A fruit shake costs R2,00 more than a chocolate milkshake. If three fruit shakes and 5 chocolate milkshakes cost R78,00, determine the individual prices.

Answer

Step 1 : Summarise the information in a table

	Price	number	Total
Fruit	$x+2$	3	$3(x+2)$
Chocolate	x	5	$5x$

Step 2 : Set up an algebraic equation

$$3(x+2)+5x = 78$$

Step 3 : Solve the equation

$$\begin{aligned} 3x+6+5x &= 78 \\ 8x &= 72 \\ x &= 9 \end{aligned}$$

Step 4 : Present the final answer

Chocolate milkshake costs R 9,00 and the Fruitshake costs R 11,00

Worked Example 45: Mathematical Modelling: Two variables

Question: Three rulers and two pens cost R 21,00. One ruler and one pen cost R 8,00. Find the cost of one ruler and one pen

Answer

Step 1 : Translate the problem using variables

Let the cost of one ruler be x rand and the cost of one pen be y rand.

Step 2 : Rewrite the information in terms of the variables

$$3x+2y = 21 \quad (10.4)$$
$$x+y = 8 \quad (10.5)$$

Step 3 : Solve the equations simultaneously

First solve the second equation for y:

$$y = 8 - x$$

and substitute the result into the first equation:

$$\begin{aligned} 3x+2(8-x) &= 21 \\ 3x+16-2x &= 21 \\ x &= 5 \end{aligned}$$

therefore

$$\begin{aligned} y &= 8-5 \\ y &= 3 \end{aligned}$$

Step 4 : Present the final answers

one Ruler costs R 5,00 and one Pen costs R 3,00

?

Exercise: Mathematical Models

1. Stephen has 1 l of a mixture containing 69% of salt. How much water must Stephen add to make the mixture 50% salt? Write your answer as a fraction.
2. The diagonal of a rectangle is 25 cm more than its width. The length of the rectangle is 17 cm more than its width. What are the dimensions of the rectangle?
3. The sum of 27 and 12 is 73 more than an unknown number. Find the unknown number.
4. The two smaller angles in a right-angled triangle are in the ratio of 1:2. What are the sizes of the two angles?
5. George owns a bakery that specialises in wedding cakes. For each wedding cake, it costs George R150 for ingredients, R50 for overhead, and R5 for advertising. George's wedding cakes cost R400 each. As a percentage of George's costs, how much profit does he make for each cake sold?
6. If 4 times a number is increased by 7, the result is 15 less than the square of the number. Find the numbers that satisfy this statement, by formulating an equation and then solving it.
7. The length of a rectangle is 2 cm more than the width of the rectangle. The perimeter of the rectangle is 20 cm. Find the length and the width of the rectangle.

10.7.4 End of Chapter Exercises

1. What are the roots of the quadratic equation $x^2 - 3x + 2 = 0$?
2. What are the solutions to the equation $x^2 + x = 6$?
3. In the equation $y = 2x^2 - 5x - 18$, which is a value of x when $y = 0$?
4. Manuel has 5 more CDs than Pedro has. Bob has twice as many CDs as Manuel has. Altogether the boys have 63 CDs. Find how many CDs each person has.
5. Seven-eighths of a certain number is 5 more than one-third of the number. Find the number.
6. A man runs to a telephone and back in 15 minutes. His speed on the way to the telephone is 5 m/s and his speed on the way back is 4 m/s. Find the distance to the telephone.
7. Solve the inequality and then answer the questions:
   $\frac{x}{3} - 14 > 14 - \frac{x}{4}$
   (a) If $x \epsilon R$, write the solution in interval notation.
   (b) if $x \epsilon Z$ and $x < 51$, write the solution as a set of integers.
8. Solve for a: $\frac{1-a}{2} - \frac{2-a}{3} > 1$
9. Solve for x: $x - 1 = \frac{42}{x}$
10. Solve for x and y: $7x + 3y = 13$ and $2x - 3y = -4$

10.8 Introduction to Functions and Graphs

Functions are mathematical building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping aeroplanes in the air. Functions can take input from many variables, but always give the same answer, unique to that function. It is the fact that you always get the same answer from a set of inputs, which is what makes functions special.

A major advantage of functions is that they allow us to *visualise* equations in terms of a *graph*. A graph is an accurate drawing of a function and is much easier to read than lists of numbers. In this chapter we will learn how to understand and create real valued functions, how to read graphs and how to draw them.

Despite their use in the problems facing humanity, functions also appear on a day-to-day level, so they are worth learning about. A function is always *dependent* on one or more variables, like time, distance or a more abstract quantity.

10.9 Functions and Graphs in the Real-World

Some typical examples of functions you may already have met include:-

- how much money you have, as a function of time. You never have more than one amount of money at any time because you can always add everything to give one number. By understanding how your money changes over time, you can plan to spend your money sensibly. Businesses find it very useful to *plot the graph* of their money over time so that they can see when they are spending too much. Such observations are not always obvious from looking at the numbers alone.
- the temperature is a very complicated function because it has so many inputs, including; the time of day, the season, the amount of clouds in the sky, the strength of the wind, where you are and many more. But the important thing is that there is only one temperature when you measure it. By understanding how the temperature is effected by these things, you can plan for the day.
- where you are is a function of time, because you cannot be in two places at once! If you were to *plot the graphs* of where two people are as a function of time, if the lines cross it means that the two people meet each other at that time. This idea is used in *logistics*, an area of mathematics that tries to plan where people and items are for businesses.
- your weight is a function of how much you eat and how much exercise you do, but everybody has a different function so that is why people are all different sizes.

10.10 Recap

The following should be familiar.

10.10.1 Variables and Constants

In section 2.4 (page 8), we were introduced to variables and constants. To recap, a *variable* can take any value in some set of numbers, so long is the equation is consistent. Most often, a variable will be written as a letter.

A *constant* has a fixed value. The number 1 is a constant. Sometimes letters are used to represent constants, as its easier to work with.

Activity :: Investigation : Variables and Constants
In the following expressions, identify the variables and the constants:

1. $2x^2 = 1$
2. $3x + 4y = 7$
3. $y = \frac{-5}{x}$
4. $y = 7^x - 2$

10.10.2 Relations and Functions

In earlier grades, you saw that variables can be *related* to each other. For example, Alan is two years older than Nathan. Therefore the relationship between the ages of Alan and Nathan can be written as $A = N + 2$, where A is Alan's age and N is Nathan's age.

In general, a *relation* is an equation which relates two variables. For example, $y = 5x$ and $y^2 + x^2 = 5$ are relations. In both examples x and y are variables and 5 is a constant, but for a given value of x the value of y will be very different in each relation.

Besides writing relations as equations, they can also be represented as words, tables and graphs. Instead of writing $y = 5x$, we could also say "y is always five times as big as x". We could also give the following table:

x	$y = 5x$
2	10
6	30
8	40
13	65
15	75

Activity :: Investigation : Relations and Functions
Complete the following table for the given functions:

x	$y = x$	$y = 2x$	$y = x + 2$
1			
2			
3			
50			
100			

10.10.3 The Cartesian Plane

When working with real valued functions, our major tool is drawing graphs. In the first place, if we have two real variables, x and y, then we can assign values to them simultaneously. That is, we can say "let x be 5 and y be 3". Just as we write "let $x = 5$" for "let x be 5", we have the shorthand notation "let $(x, y) = (5, 3)$" for "let x be 5 and y be 3". We usually think of the real numbers as an infinitely long line, and picking a number as putting a dot on that line. If we want to pick *two* numbers at the same time, we can do something similar, but now we must use two dimensions. What we do is use two lines, one for x and one for y, and rotate the one for y, as in Figure 10.1. We call this the *Cartesian plane*.

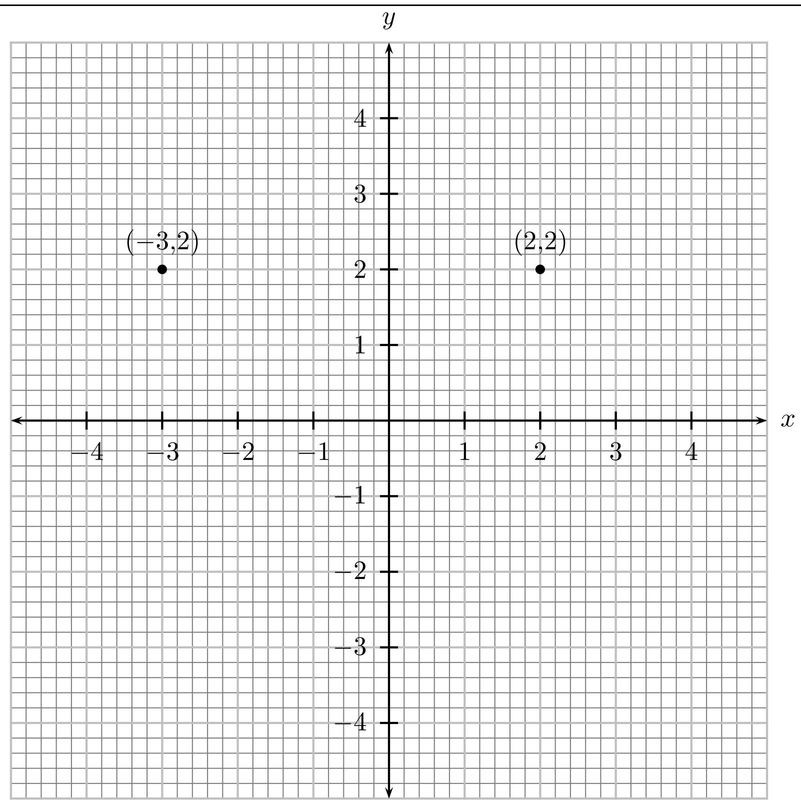


Figure 10.1: The Cartesian plane is made up of an $x-$axis (horizontal) and a $y-$axis (vertical).

10.10.4 Drawing Graphs

In order to draw the graph of a function, we need to calculate a few points. Then we plot the points on the Cartesian Plane and join the points with a smooth line.

The great beauty of doing this is that it allows us to "draw" functions, in a very abstract way. Assume that we were investigating the properties of the function $f(x) = 2x$. We could then consider all the points (x, y) such that $y = f(x)$, i.e. $y = 2x$. For example, $(1, 2), (2.5, 5)$, and $(3, 6)$ would all be such points, whereas $(3, 5)$ would not since $5 \neq 2 \times 3$. If we put a dot at each of those points, and then at every similar one for all possible values of x, we would obtain the graph shown in

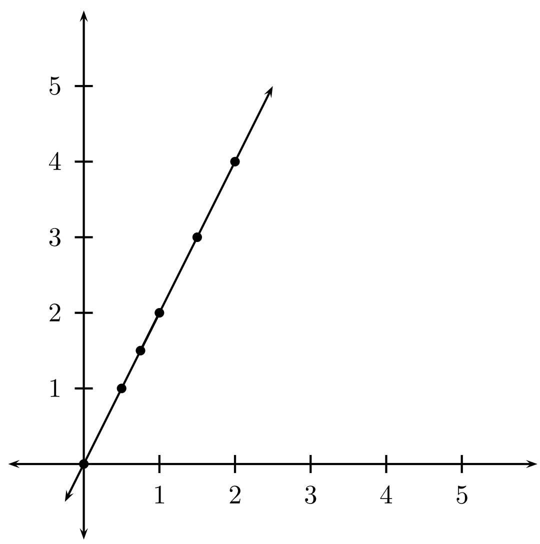


Figure 10.2: Graph of $f(x) = 2x$

The form of this graph is very pleasing – it is a simple straight line through the middle of the plane. The technique of "plotting", which we have followed here, is *the* key element in understanding functions.

Activity :: Investigation : Drawing Graphs and the Cartesian Plane
Plot the following points and draw a smooth line through them. (-6; -8),(-2; 0), (2; 8), (6; 16)

10.10.5 Notation used for Functions

Thus far you would have seen that we can use $y = 2x$ to represent a function. This notation however gets confusing when you are working with more than one function. A more general form of writing a function is to write the function as $f(x)$, where f is the function name and x is the independent variable. For example, $f(x) = 2x$ and $g(t) = 2t + 1$ are two functions.

Both notations will be used in this book.

Worked Example 46: Function notation

Question: If $f(n) = n^2 - 6n + 9$, find $f(k - 1)$ in terms of k.
Answer
Step 1 : Replace n with $k - 1$

$$\begin{aligned} f(n) &= n^2 - 6n + 9 \\ f(k-1) &= (k-1)^2 - 6(k-1) + 9 \end{aligned}$$

Step 2 : Remove brackets on RHS and simplify

$$\begin{aligned} &= k^2 - 2k + 1 - 6k + 6 + 9 \\ &= k^2 - 8k + 16 \end{aligned}$$

Worked Example 47: Function notation

Question: If $f(x) = x^2 - 4$, calculate b if $f(b) = 45$.
Answer
Step 1 : Replace x with b

$$\begin{aligned} f(b) &= b^2 - 4 \\ but f(b) &= 45 \end{aligned}$$

Step 2 : f(b) = f(b)

$$\begin{aligned} b^2 - 4 &= 45 \\ b^2 - 49 &= 0 \\ b &= +7 or - 7 \end{aligned}$$

{ExerciseRecap

1. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	1	2	3	40	50	600	700	800	900	1000

2. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	2	4	6	80	100	1200	1400	1600	1800	2000

3. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	10	20	30	400	500	6000	7000	8000	9000	10000

4. On a Cartesian plane, plot the following points: (1,2), (2,4), (3,6), (4,8), (5,10). Join the points. Do you get a straight-line?

5. If $f(x) = x + x^2$, write out:

   (a) $f(t)$
   (b) $f(a)$
   (c) $f(1)$
   (d) $f(3)$

6. If $g(x) = x$ and $f(x) = 2x$, write out:

   (a) $f(t) + g(t)$
   (b) $f(a) - g(a)$
   (c) $f(1) + g(2)$
   (d) $f(3) + g(s)$

7. A car drives by you on a straight highway. The car is travelling 10 m every second. Complete the table below by filling in how far the car has travelled away from you after 5, 10 and 20 seconds.

Time (s)	0	1	2	5	10	20
Distance (m)	0	10	20			

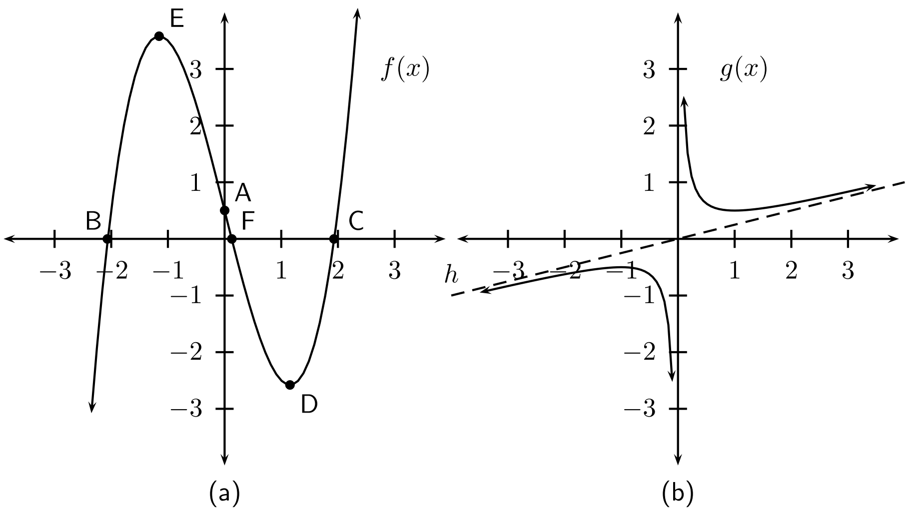
   Use the values in the table and draw a graph of distance on the y-axis and time on the x-axis.

10.11 Characteristics of Functions - All Grades

There are many characteristics of graphs that help describe the graph of any function. These properties are:

1. dependent and independent variables
2. domain and range
3. intercepts with axes
4. turning points
5. asymptotes
6. lines of symmetry
7. intervals on which the function increases/decreases
8. continuous nature of the function

Some of these words may be unfamiliar to you, but each will be clearly described. Examples of these properties are shown in Figure 10.3.



A	y-intercept
B, C, F	x-intercept
D, E	turning points

Figure 10.3: (a) Example graphs showing the characteristics of a function. (b) Example graph showing asymptotes of a function.

10.11.1 Dependent and Independent Variables

Thus far, all the graphs you have drawn have needed two values, an x-value and a y-value. The y-value is usually determined from some relation based on a given or chosen x-value. These values are given special names in mathematics. The given or chosen x-value is known as the *independent* variable, because its value can be chosen freely. The calculated y-value is known as the *dependent* variable, because its value depends on the chosen x-value.

10.11.2 Domain and Range

The *domain* of a relation is the set of all the x values for which there exists at least one y value according to that relation. The *range* is the set of all the y values, which can be obtained using at least one x value.

If the relation is of height to people, then the domain is all living people, while the range would be about 0.1 to 3 metres — no living person can have a height of 0m, and while strictly its not impossible to be taller than 3 metres, no one alive is. An important aspect of this range is that it does not contain *all* the numbers between 0.1 and 3, but only six billion of them (as many as there are people).

As another example, suppose x and y are real valued variables, and we have the relation $y = 2^x$. Then for *any* value of x, there is a value of y, so the domain of this relation is the whole set of real numbers. However, we know that no matter what value of x we choose, 2^x can never be less than or equal to 0. Hence the range of this function is all the real numbers strictly greater than zero.

These are two ways of writing the domain and range of a function, *set notation* and *interval notation*. Both notations are used in mathematics, so you should be familiar with each.

Set Notation

A set of certain x values has the following form:

$$\{x : \text{conditions, more conditions}\} \tag{10.6}$$

We read this notation as "the set of all x values where all the conditions are satisfied". For example, the set of all positive real numbers can be written as $\{x : x \in \mathbb{R}, x > 0\}$ which reads as "the set of all x values where x is a real number and is greater than zero".

Interval Notation

Here we write an interval in the form '*lower bracket, lower number, comma, upper number, upper bracket*'. We can use two types of brackets, square ones $[,]$ or round ones $(,)$. A square bracket means including the number at the end of the interval whereas a round bracket means excluding the number at the end of the interval. It is important to note that this notation can only be used for all real numbers in an interval. It cannot be used to describe integers in an interval or rational numbers in an interval.

So if x is a real number greater than 2 and less than or equal to 8, then x is any number in the interval

$$(2{,}8] \tag{10.7}$$

It is obvious that 2 is the lower number and 8 the upper number. The round bracket means 'excluding 2', since x is greater than 2, and the square bracket means 'including 8' as x is less than or equal to 8.

10.11.3 Intercepts with the Axes

The *intercept* is the point at which a graph intersects an axis. The x-intercepts are the points at which the graph cuts the x-axis and the y-intercepts are the points at which the graph cuts the y-axis.

In Figure 10.3(a), the A is the y-intercept and B, C and F are x-intercepts.

You **will** usually need to calculate the intercepts. The two most important things to remember is that at the x-intercept, $y = 0$ and at the y-intercept, $x = 0$.

For example, calculate the intercepts of $y = 3x + 5$. For the y-intercept, $x = 0$. Therefore the y-intercept is $y_{int} = 3(0) + 5 = 5$. For the x-intercept, $y = 0$. Therefore the x-intercept is found from $0 = 3x_{int} + 5$, giving $x_{int} = -\frac{5}{3}$.

10.11.4 Turning Points

Turning points only occur for graphs of functions that whose highest power is greater than 1. For example, graphs of the following functions will have turning points.

$$\begin{aligned} f(x) &= 2x^2 - 2 \\ g(x) &= x^3 - 2x^2 + x - 2 \\ h(x) &= \frac{2}{3}x^4 - 2 \end{aligned}$$

There are two types of turning points: a minimal turning point and a maximal turning point. A minimal turning point is a point on the graph where the graph stops decreasing in value and starts increasing in value and a maximal turning point is a point on the graph where the graph stops increasing in value and starts decreasing. These are shown in Figure 10.4.

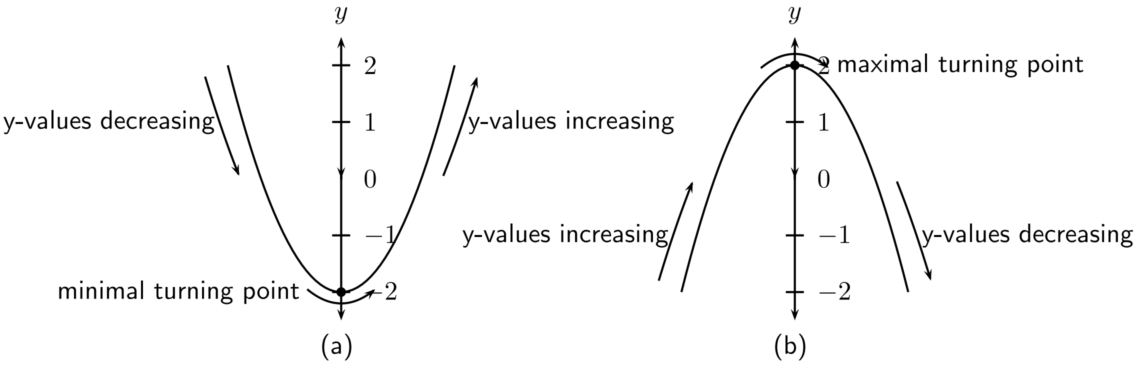


Figure 10.4: (a) Maximal turning point. (b) Minimal turning point.

In Figure 10.3(a), E is a maximal turning point and D is a minimal turning point.

10.11.5 Asymptotes

An asymptote is a straight or curved line, which the graph of a function will approach, but never touch.

In Figure 10.3(b), the y-axis and line h are both asymptotes as the graph approaches both these lines, but never touches them.

10.11.6 Lines of Symmetry

Graphs look the same on either side of lines of symmetry. These lines include the x- and y-axes. For example, in Figure 10.5 is symmetric about the y-axis. This is described as the axis of symmetry.

10.11.7 Intervals on which the Function Increases/Decreases

In the discussion of turning points, we saw that the graph of a function can start or stop increasing or decreasing at a turning point. If the graph in Figure 10.3(a) is examined, we find that the values of the graph increase and decrease over different intervals. We see that the graph increases (i.e. that the y-values increase) from -∞ to point E, then it decreases (i.e. the y-values decrease) from point E to point D and then it increases from point D to $+\infty$.

10.11.8 Discrete or Continuous Nature of the Graph

A graph is said to be continuous if there are no breaks in the graph. For example, the graph in Figure 10.3(a) can be described as a continuous graph, while the graph in Figure 10.3(b) has a

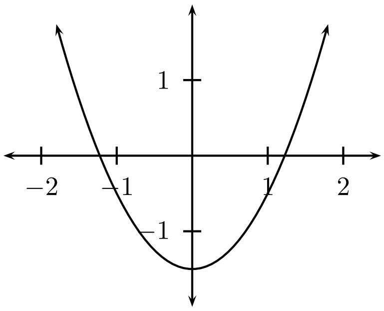


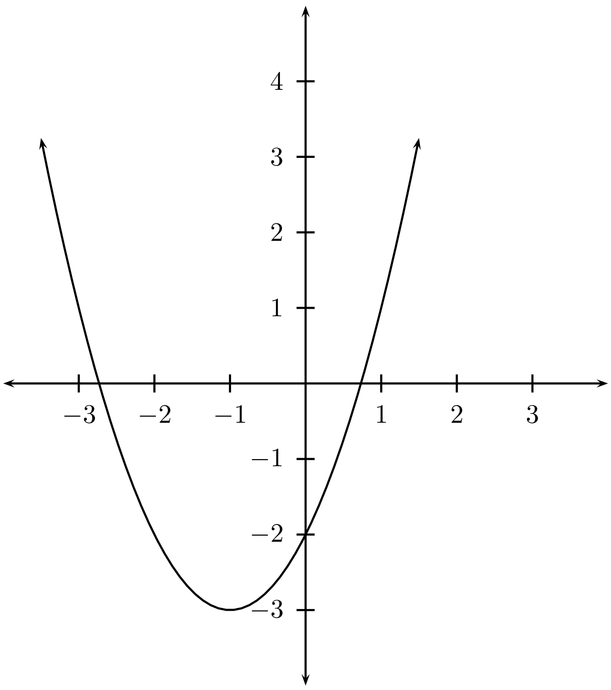
Figure 10.5: Demonstration of axis of symmetry. The y-axis is an axis of symmetry, because the graph looks the same on both sides of the y-axis.

break around the asymptotes. In Figure 10.3(b), it is clear that the graph does have a break in it around the asymptote.

?

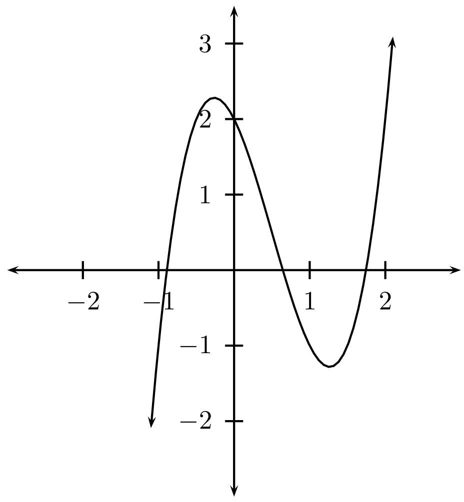
Exercise: Domain and Range

1. The domain of the function $f(x) = 2x+5$ is -3; -3; -3; 0. Determine the range of f.
2. If $g(x) = -x^2 + 5$ and x is between - 3 and 3, determine:
   (a) the domain of $g(x)$
   (b) the range of $g(x)$
3. Label, on the following graph:
   (a) the x-intercept(s)
   (b) the y-intercept(s)
   (c) regions where the graph is increasing
   (d) regions where the graph is decreasing



4. Label, on the following graph:
   (a) the x-intercept(s)
   (b) the y-intercept(s)

(c) regions where the graph is increasing
(d) regions where the graph is decreasing



10.12 Graphs of Functions

10.12.1 Functions of the form $y = ax + q$

Functions with a general form of $y = ax + q$ are called *straight line* functions. In the equation, $y = ax + q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 10.6 for the function $f(x) = 2x + 3$.

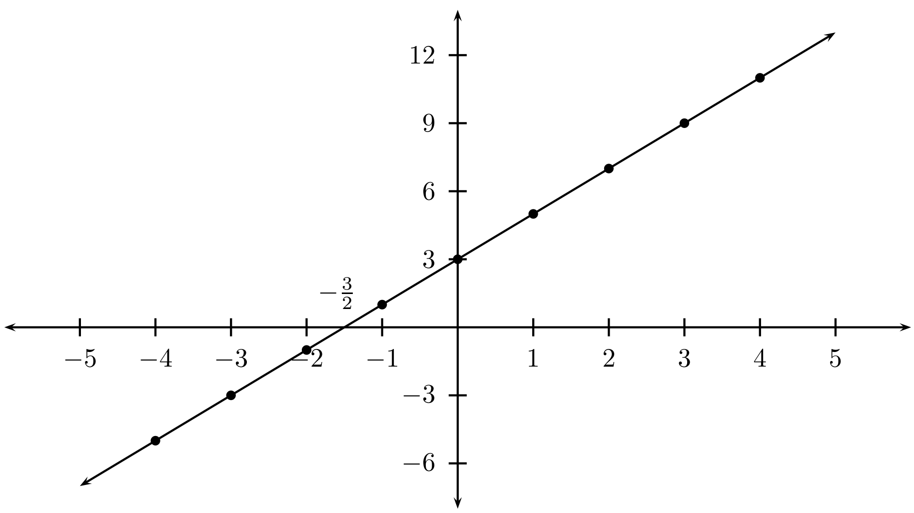


Figure 10.6: Graph of $f(x) = 2x + 3$

Activity :: Investigation : Functions of the Form $y = ax + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = x - 2$

(b) $b(x) = x - 1$
(c) $c(x) = x$
(d) $d(x) = x + 1$
(e) $e(x) = x + 2$

Use your results to deduce the effect of q.

2. On the same set of axes, plot the following graphs:
(a) $f(x) = -2 \cdot x$
(b) $g(x) = -1 \cdot x$
(c) $h(x) = 0 \cdot x$
(d) $j(x) = 1 \cdot x$
(e) $k(x) = 2 \cdot x$

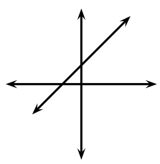
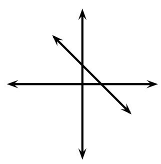
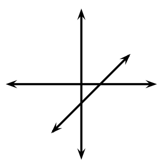
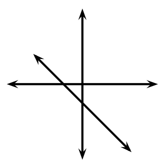
Use your results to deduce the effect of a.

You should have found that the value of a affects the slope of the graph. As a increases, the slope of the graph increases. If $a > 0$ then the graph increases from left to right (slopes upwards). If $a < 0$ then the graph increases from right to left (slopes downwards). For this reason, a is referred to as the *slope* or *gradient* of a straight-line function.

You should have also found that the value of q affects where the graph passes through the y-axis. For this reason, q is known as the *y-intercept*.

These different properties are summarised in Table 10.1.

Table 10.1: Table summarising general shapes and positions of graphs of functions of the form $y = ax + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(x) = ax + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = ax + q$ is also $\{f(x) : f(x) \in \mathbb{R}\}$ because there is no value of $f(x) \in \mathbb{R}$ for which $f(x)$ is undefined.

For example, the domain of $g(x) = x - 1$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ is $\{g(x) : g(x) \in \mathbb{R}\}$.

Intercepts

For functions of the form, $y = ax + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ax + q && (10.8)\\ y_{int} &= a(0) + q && (10.9)\\ &= q && (10.10)\end{aligned}$$

For example, the y-intercept of $g(x) = x - 1$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= x - 1\\ y_{int} &= 0 - 1\\ &= -1\end{aligned}$$

The x-intercepts are calculated as follows:

$$\begin{aligned} y &= ax + q && (10.11)\\ 0 &= a \cdot x_{int} + q && (10.12)\\ a \cdot x_{int} &= -q && (10.13)\\ x_{int} &= -\frac{q}{a} && (10.14)\end{aligned}$$

For example, the x-intercepts of $g(x) = x - 1$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= x - 1\\ 0 &= x_{int} - 1\\ x_{int} &= 1\end{aligned}$$

Turning Points

The graphs of straight line functions do not have any turning points.

Axes of Symmetry

The graphs of straight-line functions do not, generally, have any axes of symmetry.

Sketching Graphs of the Form $f(x) = ax + q$

In order to sketch graphs of the form, $f(x) = ax + q$, we need to determine three characteristics:

1. sign of a
2. y-intercept
3. x-intercept

Only two points are needed to plot a straight line graph. The easiest points to use are the x-*intercept* (where the line cuts the x-axis) and the y-intercept.

For example, sketch the graph of $g(x) = x - 1$. Mark the intercepts.

Firstly, we determine that $a > 0$. This means that the graph will have an upward slope.

The y-intercept is obtained by setting $x = 0$ and was calculated earlier to be $y_{int} = -1$. The x-intercept is obtained by setting $y = 0$ and was calculated earlier to be $x_{int} = 1$.

Worked Example 48: Drawing a straight line graph

Question: Draw the graph of $y = 2x + 2$

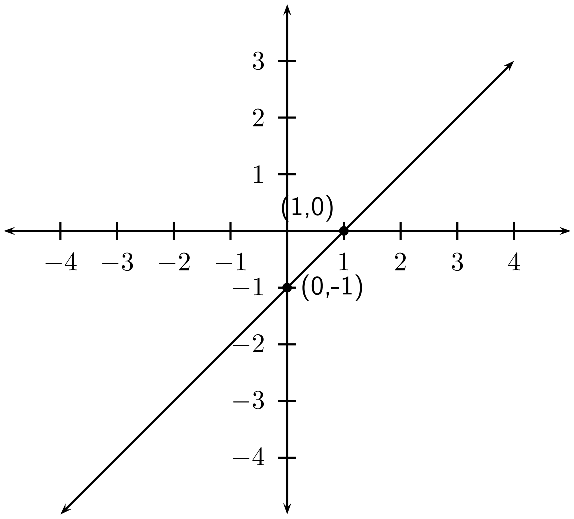


Figure 10.7: Graph of the function $g(x) = x - 1$

Answer

Step 1 : Find the y-intercept

For the intercept on the y-axis, let $x = 0$

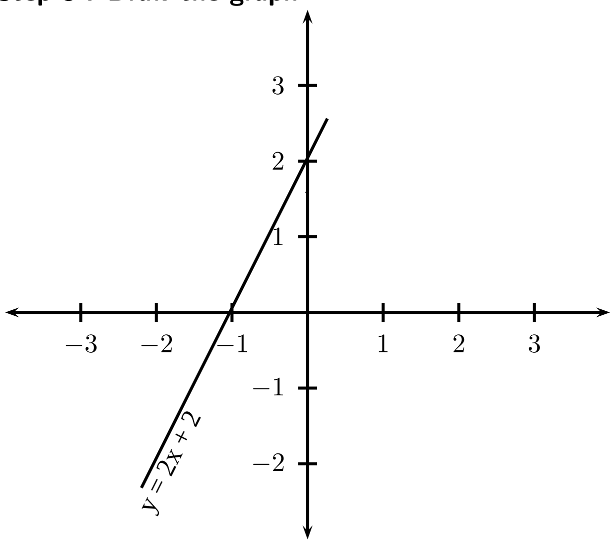
$$\begin{aligned} y &= 2(0) + 2 \\ &= 2 \end{aligned}$$

Step 2 : Find the x-intercept

For the intercept on the x-axis, let $y = 0$

$$\begin{aligned} 0 &= 2x + 2 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Step 3 : Draw the graph



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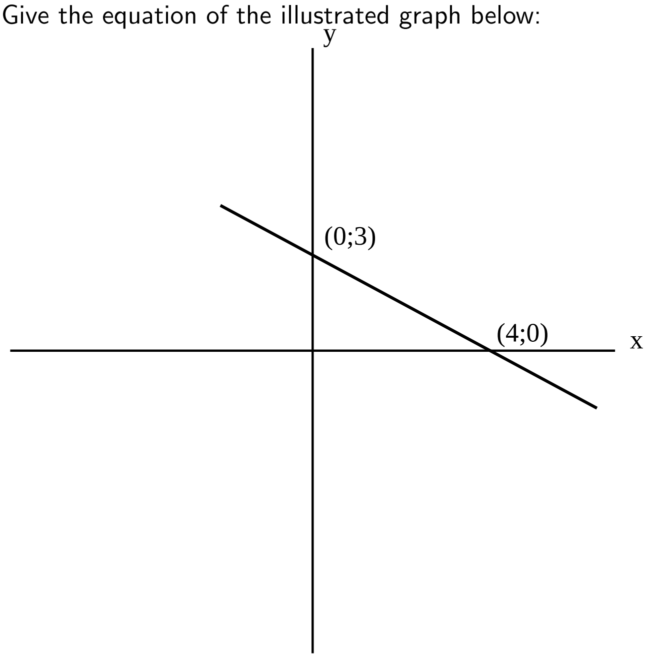
Exercise: Intercepts

1. List the y-intercepts for the following straight-line graphs:
   (a) $y = x$
   (b) $y = x - 1$
   (c) $y = 2x - 1$
   (d) $y + 1 = 2x$
2. Give the equation of the illustrated graph below:



3. Sketch the following relations on the same set of axes, clearly indicating the intercepts with the axes as well as the co-ordinates of the point of interception on the graph: $x + 2y - 5 = 0$ and $3x - y - 1 = 0$

10.12.2 Functions of the Form $y = ax^2 + q$

The general shape and position of the graph of the function of the form $f(x) = ax^2 + q$ is shown in Figure 10.8.

Activity :: Investigation : Functions of the Form $y = ax^2 + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2 \cdot x^2 + 1$
   (b) $b(x) = -1 \cdot x^2 + 1$
   (c) $c(x) = 0 \cdot x^2 + 1$
   (d) $d(x) = 1 \cdot x^2 + 1$
   (e) $e(x) = 2 \cdot x^2 + 1$

   Use your results to deduce the effect of a.
2. On the same set of axes, plot the following graphs:
   (a) $f(x) = x^2 - 2$
   (b) $g(x) = x^2 - 1$

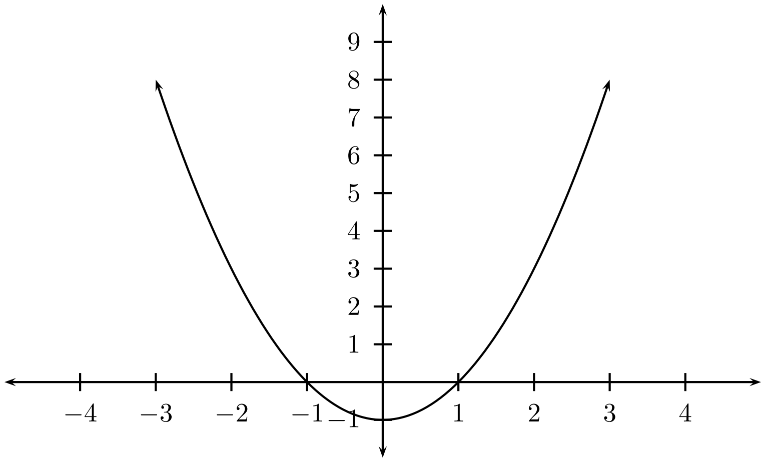


Figure 10.8: Graph of the $f(x) = x^2 - 1$.

(c) $h(x) = x^2 + 0$

(d) $j(x) = x^2 + 1$

(e) $k(x) = x^2 + 2$

Use your results to deduce the effect of q.

Complete the following table of values for the functions a to k to help with drawing the required graphs in this activity:

x	-2	-1	0	1	2
$a(x)$					
$b(x)$					
$c(x)$					
$d(x)$					
$e(x)$					
$f(x)$					
$g(x)$					
$h(x)$					
$j(x)$					
$k(x)$					

From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This is shown in Figure 10.9.

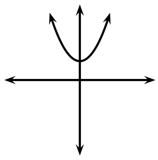
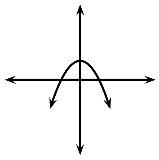
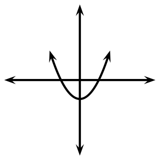
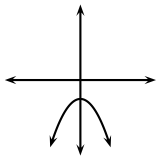
$a > 0$ (a positive smile) $a < 0$ (a negative frown)

Figure 10.9: Distinctive shape of graphs of a parabola if $a > 0$ and $a < 0$.

You should have also found that the value of q affects whether the turning point is to the left of the y-axis ($q > 0$) or to the right of the y-axis ($q < 0$).

These different properties are summarised in Table **??**.

Table 10.2: Table summarising general shapes and positions of functions of the form $y = ax^2 + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(x) = ax^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = ax^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{aligned} x^2 &\geq 0 \quad \text{(The square of an expression is always positive)} \\ ax^2 &\geq 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\ ax^2 + q &\geq q \\ f(x) &\geq q \end{aligned}$$

This tells us that for all values of x, $f(x)$ is always greater than q. Therefore if $a > 0$, the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$. This is left as an exercise.

For example, the domain of $g(x) = x^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} x^2 &\geq 0 \\ x^2 + 2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Intercepts

For functions of the form, $y = ax^2 + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ax^2 + q && (10.15) \\ y_{int} &= a(0)^2 + q && (10.16) \\ &= q && (10.17) \end{aligned}$$

For example, the y-intercept of $g(x) = x^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= x^2 + 2 \\ y_{int} &= 0^2 + 2 \\ &= 2 \end{aligned}$$

The x-intercepts are calculated as follows:

$$y = ax^2 + q \quad (10.18)$$
$$0 = ax_{int}^2 + q \quad (10.19)$$
$$ax_{int}^2 = -q \quad (10.20)$$
$$x_{int} = \pm\sqrt{-\frac{q}{a}} \quad (10.21)$$

However, (10.21) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis.

For example, the x-intercepts of $g(x) = x^2 + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= x^2 + 2 \\ 0 &= x_{int}^2 + 2 \\ -2 &= x_{int}^2 \end{aligned}$$

which is not real. Therefore, the graph of $g(x) = x^2 + 2$ does not have any x-intercepts.

Turning Points

The turning point of the function of the form $f(x) = ax^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$ and if $a < 0$ then the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{aligned} q &= ax_{tp}^2 + q \\ 0 &= ax_{tp}^2 \\ 0 &= x_{tp}^2 \\ x_{tp} &= 0 \end{aligned}$$

$\therefore x = 0$ at $f(x) = q$. The co-ordinates of the (minimal) turning point is therefore $(0; q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point is $(0; q)$.

Axes of Symmetry

There is one axis of symmetry for the function of the form $f(x) = ax^2 + q$ that passes through the turning point. Since the turning point lies on the y-axis, the axis of symmetry is the y-axis.

Sketching Graphs of the Form $f(x) = ax^2 + q$

In order to sketch graphs of the form, $f(x) = ax^2 + q$, we need to calculate determine four characteristics:

1. sign of a

2. domain and range
3. turning point
4. y-intercept
5. x-intercept

For example, sketch the graph of $g(x) = -\frac{1}{2}x^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{aligned} x^2 &\geq 0 \\ -\frac{1}{2}x^2 &\leq 0 \\ -\frac{1}{2}x^2 - 3 &\leq -3 \\ \therefore f(x) &\leq -3 \end{aligned}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{aligned} -\frac{1}{2}x^2 - 3 &= -3 \\ -\frac{1}{2}x^2 - 3 + 3 &= 0 \\ -\frac{1}{2}x^2 &= 0 \\ \text{Divide both sides by } -\tfrac{1}{2}\text{:} \quad x^2 &= 0 \\ \text{Take square root of both sides:} \quad x &= 0 \\ \therefore \quad x &= 0 \end{aligned}$$

The coordinates of the turning point are: $(0, -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{aligned} y_{int} &= -\frac{1}{2}(0)^2 - 3 \\ &= -\frac{1}{2}(0) - 3 \\ &= -3 \end{aligned}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{aligned} 0 &= -\frac{1}{2}x_{int}^2 - 3 \\ 3 &= -\frac{1}{2}x_{int}^2 \\ -3 \cdot 2 &= x_{int}^2 \\ -6 &= x_{int}^2 \end{aligned}$$

which is not real. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is the y-axis.

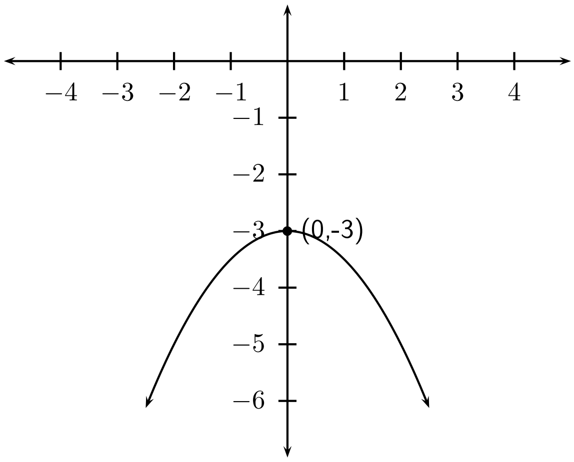
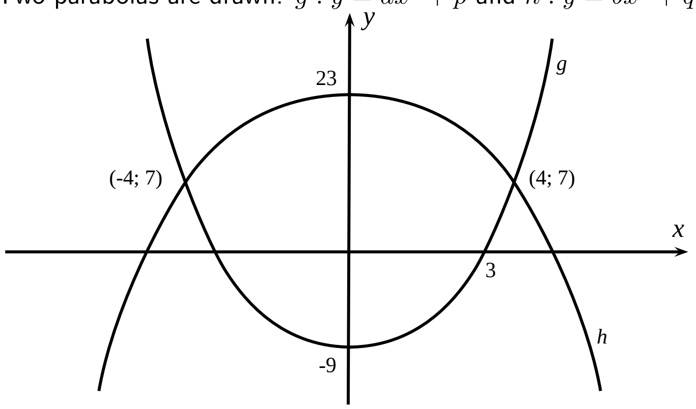


Figure 10.10: Graph of the function $f(x) = -\frac{1}{2}x^2 - 3$

?

Exercise: Parabolas

1. Show that if $a < 0$ that the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty, q]\}$.
2. Draw the graph of the function $y = -x^2 + 4$ showing all intercepts with the axes.
3. Two parabolas are drawn: $g : y = ax^2 + p$ and $h : y = bx^2 + q$.



   (a) Find the values of a and p.
   (b) Find the values of b and q.
   (c) Find the values of x for which $ax^2 + p \geq bx^2 + q$.
   (d) For what values of x is g increasing ?

10.12.3 Functions of the Form $y = \frac{a}{x} + q$

Functions of the form $y = \frac{a}{x} + q$ are known as *hyperbolic* functions. The general form of the graph of this function is shown in Figure 10.11.

Activity :: Investigation : Functions of the Form $y = \frac{a}{x} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = \frac{-2}{x} + 1$
   (b) $b(x) = \frac{-1}{x} + 1$

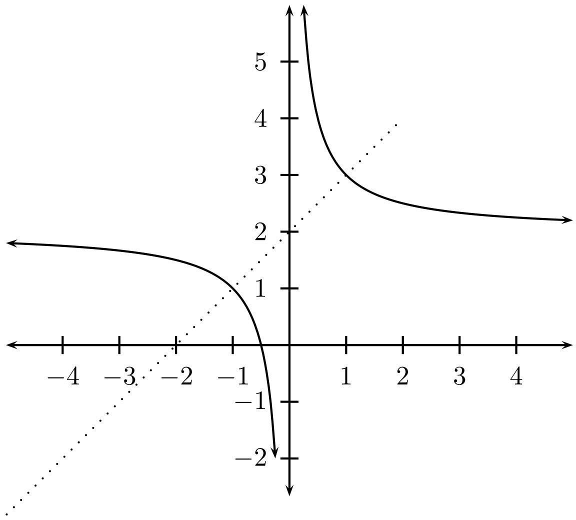


Figure 10.11: General shape and position of the graph of a function of the form $f(x) = \frac{a}{x} + q$.

(c) $c(x) = \frac{0}{x} + 1$
(d) $d(x) = \frac{+1}{x} + 1$
(e) $e(x) = \frac{+2}{x} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:
(a) $f(x) = \frac{1}{x} - 2$
(b) $g(x) = \frac{1}{x} - 1$
(c) $h(x) = \frac{1}{x} + 0$
(d) $j(x) = \frac{1}{x} + 1$
(e) $k(x) = \frac{1}{x} + 2$

Use your results to deduce the effect of q.

You should have found that the value of a affects whether the graph is located in the first and third quadrants of Cartesian plane.

You should have also found that the value of q affects whether the graph lies above the x-axis ($q > 0$) or below the x-axis ($q < 0$).

These different properties are summarised in Table 10.3. The axes of symmetry for each graph are shown as a dashed line.

Domain and Range

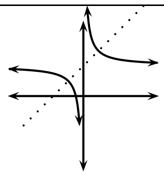
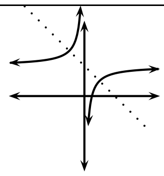
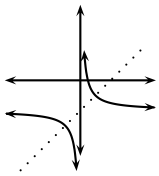
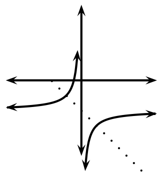
For $y = \frac{a}{x} + q$, the function is undefined for $x = 0$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq 0\}$.

We see that $y = \frac{a}{x} + q$ can be re-written as:

$$\begin{aligned} y &= \frac{a}{x} + q \\ y - q &= \frac{a}{x} \\ \text{If } x \neq 0 \text{ then: } (y - q)(x) &= a \\ x &= \frac{a}{y - q} \end{aligned}$$

This shows that the function is undefined at $y = q$. Therefore the range of $f(x) = \frac{a}{x} + q$ is $\{f(x) : f(x) \in (-\infty, q) \cup (q, \infty)\}$.

Table 10.3: Table summarising general shapes and positions of functions of the form $y = \frac{a}{x} + q$. The axes of symmetry are shown as dashed lines.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

For example, the domain of $g(x) = \frac{2}{x} + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$.

$$
\begin{aligned}
y &= \frac{2}{x} + 2 \\
(y-2) &= \frac{2}{x} \\
\text{If } x \neq 0 \text{ then: } \quad x(y-2) &= 2 \\
x &= \frac{2}{y-2}
\end{aligned}
$$

We see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

Intercepts

For functions of the form, $y = \frac{a}{x} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$
\begin{aligned}
y &= \frac{a}{x} + q && (10.22) \\
y_{int} &= \frac{a}{0} + q && (10.23)
\end{aligned}
$$

which is undefined. Therefore there is no y-intercept.

For example, the y-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$ to get:

$$
\begin{aligned}
y &= \frac{2}{x} + 2 \\
y_{int} &= \frac{2}{0} + 2
\end{aligned}
$$

which is undefined.

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= \frac{a}{x} + q && (10.24) \\ 0 &= \frac{a}{x_{int}} + q && (10.25) \\ \frac{a}{x_{int}} &= -q && (10.26) \\ a &= -q(x_{int}) && (10.27) \\ x_{int} &= \frac{a}{-q} && (10.28) \\ & && (10.29) \end{aligned}$$

For example, the x-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ 0 &= \frac{2}{x_{int}} + 2 \\ -2 &= \frac{2}{x_{int}} \\ -2(x_{int}) &= 2 \\ x_{int} &= \frac{2}{-2} \\ x_{int} &= -1 \end{aligned}$$

Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = 0$ and for $y = q$. Therefore the asymptotes are $x = 0$ and $y = q$.

For example, the domain of $g(x) = \frac{2}{x} + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = 0$ and $y = 2$.

Sketching Graphs of the Form $f(x) = \frac{a}{x} + q$

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x} + q$, we need to calculate determine four characteristics:

1. domain and range
2. asymptotes
3. y-intercept
4. x-intercept

For example, sketch the graph of $g(x) = \frac{2}{x} + 2$. Mark the intercepts and asymptotes.

We have determined the domain to be $\{x : x \in \mathbb{R}, x \neq 0\}$ and the range to be $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$. Therefore the asymptotes are at $x = 0$ and $y = 2$.

There is no y-intercept and the x-intercept is $x_{int} = -1$.

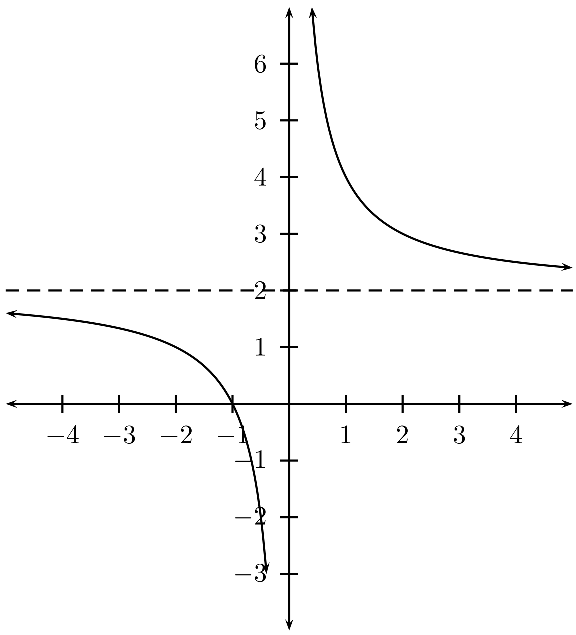


Figure 10.12: Graph of $g(x) = \frac{2}{x} + 2$.

?

Exercise: Graphs

1. Using grid paper, draw the graph of $xy = -6$.
   (a) Does the point (-2; 3) lie on the graph ? Give a reason for your answer.
   (b) Why is the point (-2; -3) not on the graph ?
   (c) If the x-value of a point on the drawn graph is $0{,}25$, what is the corresponding y-value ?
   (d) What happens to the y-values as the x-values become very large ?
   (e) With the line $y = -x$ as line of symmetry, what is the point symmetrical to (-2; 3) ?
2. Draw the graph of $xy = 8$.
   (a) How would the graph $y = \frac{8}{3} + 3$ compare with that of $xy = 8$? Explain your answer fully.
   (b) Draw the graph of $y = \frac{8}{3} + 3$ on the same set of axes.

10.12.4 Functions of the Form $y = ab^{(x)} + q$

Functions of the form $y = ab^{(x)} + q$ are known as *exponential* functions. The general shape of a graph of a function of this form is shown in Figure 10.13.

Activity :: Investigation : Functions of the Form $y = ab^{(x)} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2 \cdot b^{(x)} + 1$
   (b) $b(x) = -1 \cdot b^{(x)} + 1$
   (c) $c(x) = -0 \cdot b^{(x)} + 1$

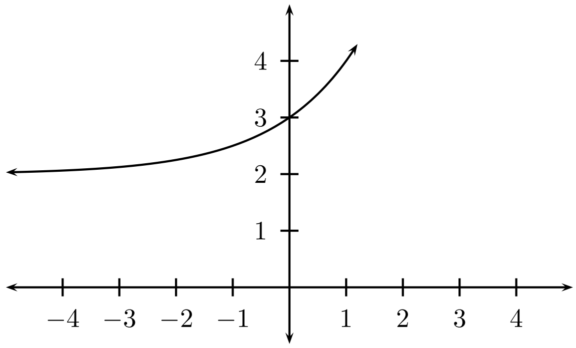


Figure 10.13: General shape and position of the graph of a function of the form $f(x) = ab^{(x)} + q$.

(d) $d(x) = -1 \cdot b^{(x)} + 1$
(e) $e(x) = -2 \cdot b^{(x)} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:
   (a) $f(x) = 1 \cdot b^{(x)} - 2$
   (b) $g(x) = 1 \cdot b^{(x)} - 1$
   (c) $h(x) = 1 \cdot b^{(x)} + 0$
   (d) $j(x) = 1 \cdot b^{(x)} + 1$
   (e) $k(x) = 1 \cdot b^{(x)} + 2$

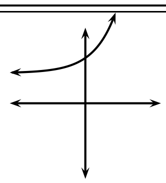
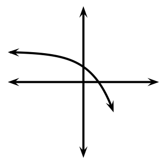
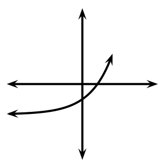
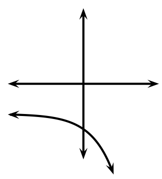
   Use your results to deduce the effect of q.

You should have found that the value of a affects whether the graph curves upwards ($a > 0$) or curves downwards ($a < 0$).

You should have also found that the value of q affects the position of the y-intercept.

These different properties are summarised in Table 10.4.

Table 10.4: Table summarising general shapes and positions of functions of the form $y = ab^{(x)} + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $y = ab^{(x)} + q$, the function is defined for all real values of x. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^{(x)} + q$ is dependent on the sign of a.

If $a > 0$ then:

$$\begin{aligned} b^{(x)} &\geq 0 \\ a \cdot b^{(x)} &\geq 0 \\ a \cdot b^{(x)} + q &\geq q \\ f(x) &\geq q \end{aligned}$$

Therefore, if $a > 0$, then the range is $\{f(x) : f(x) \in [q,\infty)\}$.

If $a < 0$ then:

$$\begin{aligned} b^{(x)} &\leq 0 \\ a \cdot b^{(x)} &\leq 0 \\ a \cdot b^{(x)} + q &\leq q \\ f(x) &\leq q \end{aligned}$$

Therefore, if $a < 0$, then the range is $\{f(x) : f(x) \in (-\infty,q]\}$.

For example, the domain of $g(x) = 3 \cdot 2^x + 2$ is $\{x : x \in \mathbb{R}\}$. For the range,

$$\begin{aligned} 2^x &\geq 0 \\ 3 \cdot 2^x &\geq 0 \\ 3 \cdot 2^x + 2 &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Intercepts

For functions of the form, $y = ab^{(x)} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ab^{(x)} + q && (10.30) \\ y_{int} &= ab^{(0)} + q && (10.31) \\ &= a(1) + q && (10.32) \\ &= a + q && (10.33) \end{aligned}$$

For example, the y-intercept of $g(x) = 3 \cdot 2^x + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^x + 2 \\ y_{int} &= 3 \cdot 2^0 + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= ab^{(x)} + q && (10.34) \\ 0 &= ab^{(x_{int})} + q && (10.35) \\ ab^{(x_{int})} &= -q && (10.36) \\ b^{(x_{int})} &= -\frac{q}{a} && (10.37) \end{aligned}$$

Which only has a real solution if either $a < 0$ or $q < 0$. Otherwise, the graph of the function of form $y = ab^{(x)} + q$ does not have any x-intercepts.

For example, the x-intercept of $g(x) = 3 \cdot 2^x + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^x + 2 \\ 0 &= 3 \cdot 2^{x_{int}} + 2 \\ -2 &= 3 \cdot 2^{x_{int}} \\ 2^{x_{int}} &= \frac{-2}{3} \end{aligned}$$

which has no real solution. Therefore, the graph of $g(x) = 3 \cdot 2^x + 2$ does not have any x-intercepts.

Asymptotes

There are two asymptotes for functions of the form $y = ab^{(x)} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = 0$ and for $y = q$. Therefore the asymptotes are $x = 0$ and $y = q$.

For example, the domain of $g(x) = 3 \cdot 2^x + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = 0$ and $y = 2$.

Sketching Graphs of the Form $f(x) = ab^{(x)} + q$

In order to sketch graphs of functions of the form, $f(x) = ab^{(x)} + q$, we need to calculate determine four characteristics:

1. domain and range
2. y-intercept
3. x-intercept

For example, sketch the graph of $g(x) = 3 \cdot 2^x + 2$. Mark the intercepts.

We have determined the domain to be $\{x : x \in \mathbb{R}\}$ and the range to be $\{g(x) : g(x) \in [2,\infty)\}$.

The y-intercept is $y_{int} = 5$ and there are no x-intercepts.

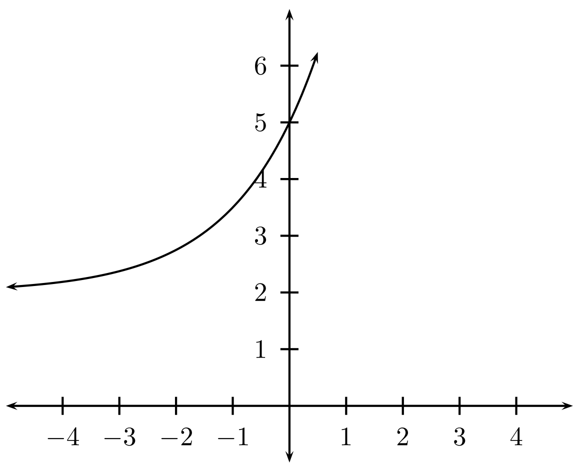
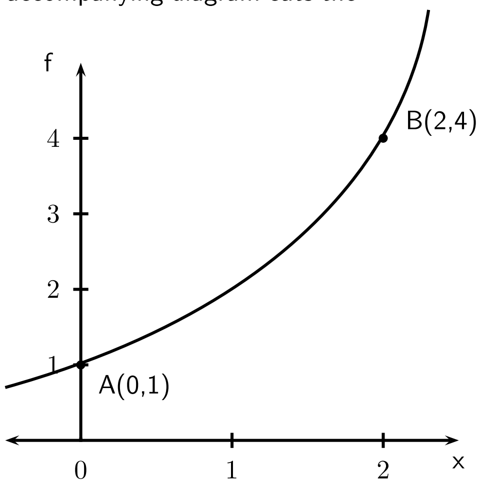


Figure 10.14: Graph of $g(x) = 3 \cdot 2^x + 2$.

?

Exercise: Exponential Functions and Graphs

1. Draw the graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$ on the same set of axes.
   (a) Is the x-axis and asymptote or and axis of symmetry to both graphs ? Explain your answer.
   (b) Which graph is represented by the equation $y = 2^{-x}$? Explain your answer.
   (c) Solve the equation $2^x = (\frac{1}{2})^x$ graphically and check that your answer is correct by using substitution.
   (d) Predict how the graph $y = 2.2^x$ will compare to $y = 2^x$ and then draw the graph of $y = 2.2^x$ on the same set of axes.
2. The curve of the exponential function f in the accompanying diagram cuts the y-axis at the point A(0; 1) and B(2; 4) is on f.



   (a) Determine the equation of the function f.
   (b) Determine the equation of h, the function of which the curve is the reflection of the curve of f in the x-axis.
   (c) Determine the range of h.

10.13 End of Chapter Exercises

1. Given the functions $f(x) = -2x^2 - 18$ and $g(x) = -2x + 6$
   (a) Draw f and g on the same set of axes.
   (b) Calculate the points of intersection of f and g.
   (c) Hence use your graphs and the points of intersection to solve for x when:
      i. $f(x) > 0$
      ii. $\frac{f(x)}{g(x)} \leq 0$
   (d) Give the equation of the reflection of f in the x-axis.
2. After a ball is dropped, the rebound height of each bounce decreases. The equation $y = 5(0.8)^x$ shows the relationship between x, the number of bounces, and y, the height of the bounce, for a certain ball. What is the approximate height of the fifth bounce of this ball to the nearest tenth of a unit ?
3. Marc had 15 coins in five rand and two rand pieces. He had 3 more R2-coins than R5-coins. He wrote a system of equations to represent this situation, letting x represent the number of five rand coins and y represent the number of two rand coins. Then he solved the system by graphing.

(a) Write down the system of equations.

(b) Draw their graphs on the same set of axes.

(c) What is the solution?

Chapter 11

Average Gradient - Grade 10 Extension

11.1 Introduction

In chapter 10.7.4, we saw that the gradient of a straight line graph is calculated as:

$$\frac{y_2 - y_1}{x_2 - x_1} \tag{11.1}$$

for two points (x_1,y_1) and (x_2,y_2) on the graph.

We can now define the *average gradient* between any two points, (x_1,y_1) and (x_2,y_2) as:

$$\frac{y_2 - y_1}{x_2 - x_1}. \tag{11.2}$$

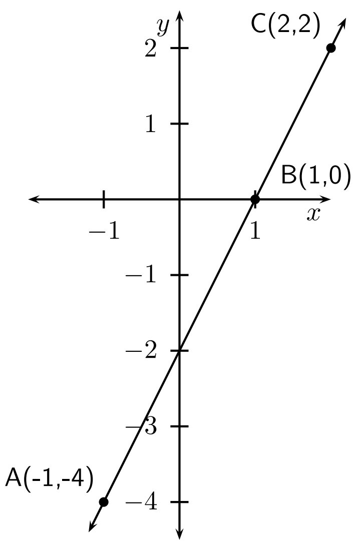
This is the same as (11.1).

11.2 Straight-Line Functions

Activity :: Investigation : Average Gradient - Straight Line Function

Fill in the table by calculating the average gradient over the indicated intervals for the function $f(x) = 2x - 2$:

	x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
A-B					
A-C					
B-C					



What do you notice about the gradients over each interval?

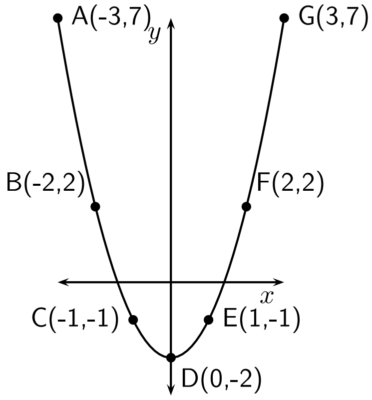
The average gradient of a straight-line function is the same over any two intervals on the function.

11.3 Parabolic Functions

Activity :: Investigation : Average Gradient - Parabolic Function

Fill in the table by calculating the average gradient over the indicated intervals for the function $f(x) = 2x - 2$:

	x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
A-B					
B-C					
C-D					
D-E					
E-F					
F-G					



What do you notice about the average gradient over each interval? What can you say about the average gradients between A and D compared to the average gradients between D and G?

The average gradient of a parabolic function depends on the interval and is the gradient of a straight line that passes through the points on the interval.

For example, in Figure 11.1 the various points have been joined by straight-lines. The average gradients between the joined points are then the gradients of the straight lines that pass through the points.

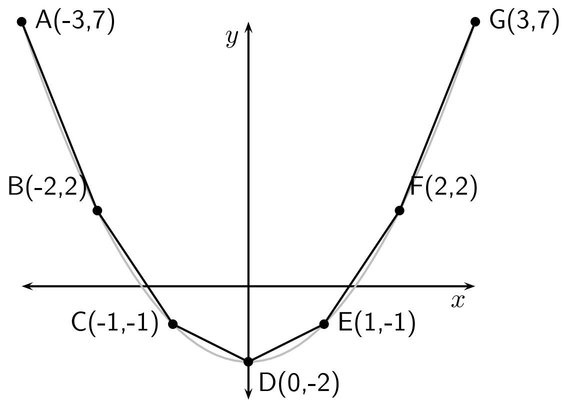


Figure 11.1: The average gradient between two points on a curve is the gradient of the straight line that passes through the points.

Method: Average Gradient

Given the equation of a curve and two points (x_1, x_2):

1. Write the equation of the curve in the form $y = \ldots$.
2. Calculate y_1 by substituting x_1 into the equation for the curve.
3. Calculate y_2 by substituting x_2 into the equation for the curve.
4. Calculate the average gradient using:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Worked Example 49: Average Gradient

Question: Find the average gradient of the curve $y = 5x^2 - 4$ between the points $x = -3$ and $x = 3$

Answer

Step 1 : Label points

Label the points as follows:

$$x_1 = -3$$

$$x_2 = 3$$

to make it easier to calculate the gradient.

Step 2 : Calculate the y coordinates

We use the equation for the curve to calculate the y-value at x_1 and x_2.

$$\begin{aligned} y_1 &= 5x_1^2 - 4 \\ &= 5(-3)^2 - 4 \\ &= 5(9) - 4 \\ &= 41 \end{aligned}$$

$$\begin{aligned} y_2 &= 5x_2^2 - 4 \\ &= 5(3)^2 - 4 \\ &= 5(9) - 4 \\ &= 41 \end{aligned}$$

Step 3 : Calculate the average gradient

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{41 - 41}{3 - (-3)} \\ &= \frac{0}{3 + 3} \\ &= \frac{0}{6} \\ &= 0 \end{aligned}$$

Step 4 : Write the final answer

The average gradient between $x = -3$ and $x = 3$ on the curve $y = 5x^2 - 4$ is 0.

11.4 End of Chapter Exercises

1. An object moves according to the function $d = 2t^2 + 1$, where d is the distance in metres and t the time in seconds. Calculate the average speed of the object between 2 and 3 seconds.
2. Given: $f(x) = x^3 - 6x$.
   Determine the average gradient between the points where $x = 1$ and $x = 4$.

Chapter 12

Geometry Basics

12.1 Introduction

The purpose of this chapter is to recap some of the ideas that you learned in geometry and trigonometry in earlier grades. You should feel comfortable with the work covered in this chapter before attempting to move onto the Grade 10 Geometry Chapter (Chapter 13) or the Grade 10 Trigonometry Chapter (Chapter 14). This chapter revises:

1. Terminology: quadrilaterals, vertices, sides, angles, parallel lines, perpendicular lines, diagonals, bisectors, transversals
2. Similarities and differences between quadrilaterals
3. Properties of triangles and quadrilaterals
4. Congruence
5. Classification of angles into acute, right, obtuse, straight, reflex or revolution
6. Theorem of Pythagoras which is used to calculate the lengths of sides of a right-angled triangle

12.2 Points and Lines

The two simplest objects in geometry are *points* and *lines*.

A point is something that is not very wide or high and is usually used in geometry as a marker of a position. Points are usually labelled with a capital letter. Some examples of points are shown in Figure 12.1.

A line is formed when many points are placed next to each other. Lines can be straight or curved, but are always continuous. This means that there are never any breaks in the lines. The endpoints of lines are labelled with capital letters. Examples of two lines are shown in Figure 12.1.

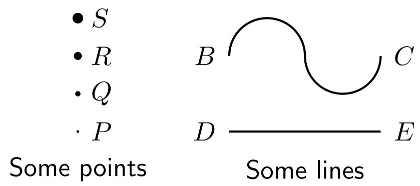


Figure 12.1: Examples of some points (labelled P, Q, R and S) and some lines (labelled BC and DE).

Lines are labelled according to the start point and end point. We call the line that starts at a point A and ends at a point B, AB. Since the line from point B to point A is the same as the line from point A to point B, we have that AB=BA.

The length of the line between points A and B is AB. So if we say $AB = CD$ we mean that the length of the line between A and B is equal to the length of the line between C and D.

In science, we sometimes talk about a *vector* and this is just a fancy way of saying the we are referring to the line that starts at one point and moves in the direction of the other point. We label a vector in a similar manner to a line, with $\vec{AB}$ referring to the vector from the point A with length AB and in the direction from point A to point B. Similarly, $\vec{BA}$ is the line segment with the same length but direction from point B to point A. Usually, vectors are only equal if they have the same length *and* same direction. So, usually, $\vec{AB} \neq \vec{BA}$.

A line is measured in *units of length*. Some common units of length are listed in Table 12.1.

Table 12.1: Some common units of length and their abbreviations.

Unit of Length	**Abbreviation**
kilometre	km
metre	m
centimetre	cm
millimetre	mm

12.3 Angles

An *angle* is formed when two straight lines meet at a point. The point at which two lines meet is known as a *vertex*. Angles are labelled with a ˆ on a letter, for example, in Figure 12.3, the angle is at $\hat{B}$. Angles can also be labelled according to the line segments that make up the angle. For example, in Figure 12.3, the angle is made up when line segments CB and BA meet. So, the angle can be referred to as $\angle CBA$ or $\angle ABC$. The $\angle$ symbol is a short method of writing angle in geometry.

Angles are measured in *degrees* which is denoted by °.

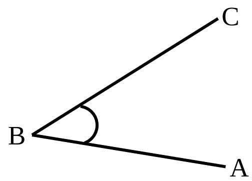


Figure 12.2: Angle labelled as $\hat{B}$, $\angle CBA$ or $\angle ABC$

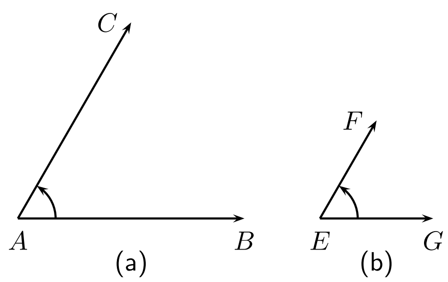


Figure 12.3: Examples of angles. $\hat{A} = \hat{E}$, even though the lines making up the angles are of different lengths.

12.3.1 Measuring angles

The size of an angle does not depend on the length of the lines that are joined to make up the angle, but depends only on how both the lines are placed as can be seen in Figure 12.3. This means that the idea of length cannot be used to measure angles. An angle is a rotation around the vertex.

Using a Protractor

A protractor is a simple tool that is used to measure angles. A picture of a protractor is shown in Figure 12.4.

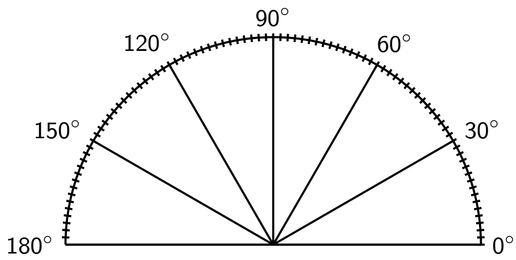


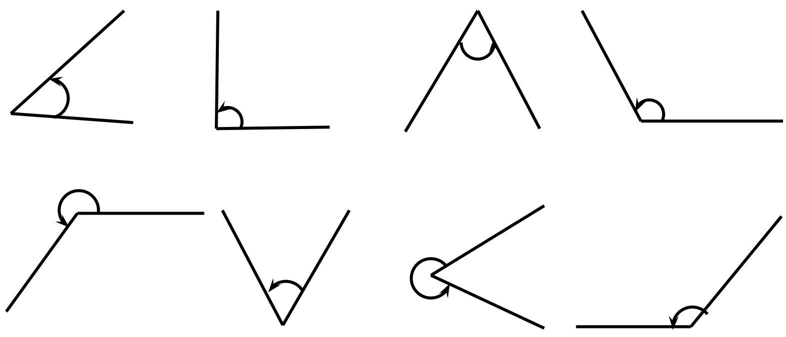
Figure 12.4: Diagram of a protractor.

Method:

Using a protractor

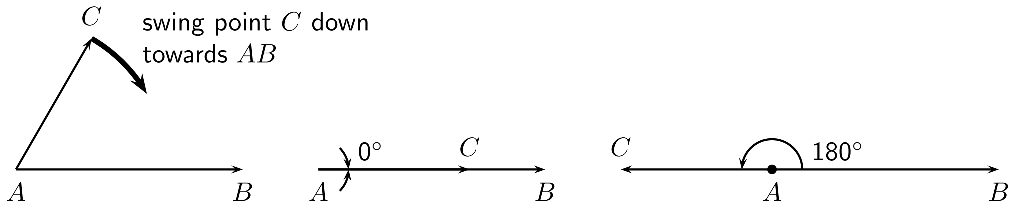
1. Place the bottom line of the protractor along one line of the angle.
2. Move the protractor along the line so that the centre point on the protractor is at the vertex of the two lines that make up the angle.
3. Follow the second line until it meets the marking on the protractor and read off the angle. Make sure you start measuring at 0°.

Activity :: Measuring Angles : Use a protractor to measure the following angles:



12.3.2 Special Angles

What is the smallest angle that can be drawn? The figure below shows two lines (CA and AB) making an angle at a common vertex A. If line CA is rotated around the common vertex A, down towards line AB, then the smallest angle that can be drawn occurs when the two lines are pointing in the same direction. This gives an angle of 0°.



If line CA is now swung upwards, any other angle can be obtained. If line CA and line AB point in opposite directions (the third case in the figure) then this forms an angle of 180°.

Important: If three points A, B and C lie on a straight line, then the angle between them is 180°. Conversely, if the angle between three points is 180°, then the points lie on a straight line.

An angle of 90° is called a *right angle*. A right angle is half the size of the angle made by a straight line (180°). We say CA is *perpendicular* to AB or $CA \perp AB$. An angle twice the size of a straight line is 360°. An angle measuring 360° looks identical to an angle of 0°, except for the labelling. We call this a *revolution*.

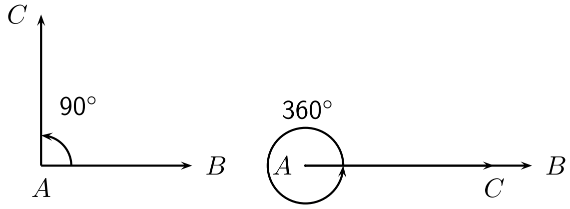


Figure 12.5: An angle of 90° is known as a *right angle*.

Extension: Angles larger than 360°

All angles larger than 360° also look like we have seen them before. If you are given an angle that is larger than 360°, continue subtracting 360° from the angle, until you get an answer that is between 0°and 360°. Angles that measure more than 360° are largely for mathematical convenience.

Important:

- *Acute angle*: An angle $\geq 0°$ and $< 90°$.
- *Right angle*: An angle measuring $90°$.
- *Obtuse angle*: An angle $> 90°$ and $< 180°$.
- *Straight angle*: An angle measuring 180°.
- *Reflex angle*: An angle $> 180°$ and $< 360°$.
- *Revolution*: An angle measuring $360°$.

These are simply labels for angles in particular ranges, shown in Figure 12.6.

Once angles can be measured, they can then be compared. For example, all right angles are 90°, therefore all right angles are equal, and an obtuse angle will always be larger than an acute angle.

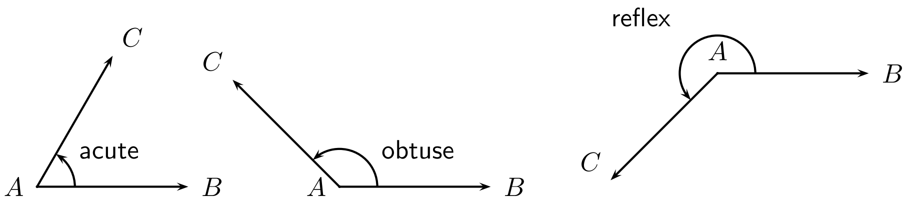


Figure 12.6: Three types of angles defined according to their ranges.

12.3.3 Special Angle Pairs

In Figure 12.7, straight lines AB and CD intersect at point X, forming four angles: $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$ and $\hat{X}_4$.

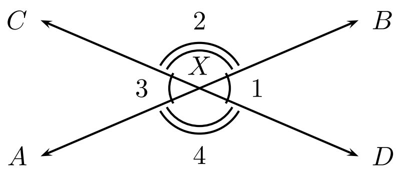


Figure 12.7: Two intersecting straight lines with vertical angles $\hat{X}_1$,$\hat{X}_3$ and $\hat{X}_2$,$\hat{X}_4$.

The table summarises the special angle pairs that result.

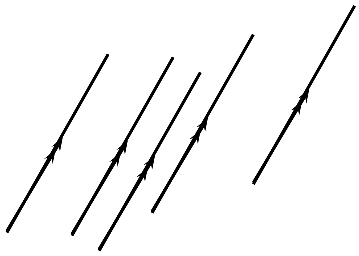
Special Angle	Property	Example
adjacent angles	share a common vertex and a common side	($\hat{X}_1$,$\hat{X}_2$), ($\hat{X}_2$,$\hat{X}_3$), ($\hat{X}_3$,$\hat{X}_4$), ($\hat{X}_4$,$\hat{X}_1$)
linear pair (adjacent angles on a straight line)	adjacent angles formed by two intersecting straight lines that by definition add to 180°	$\hat{X}_1 + \hat{X}_2 = 180°$ $\hat{X}_2 + \hat{X}_3 = 180°$ $\hat{X}_3 + \hat{X}_4 = 180°$ $\hat{X}_4 + \hat{X}_1 = 180°$
vertically opposite angles	angles formed by two intersecting straight lines that share a vertex but do not share any sides	$\hat{X}_1 = \hat{X}_3$ $\hat{X}_2 = \hat{X}_4$
supplementary angles	two angles whose sum is 180°	
complementary angles	two angles whose sum is 90°	

Important: The vertically opposite angles formed by the intersection of two straight lines are equal. Adjacent angles on a straight line are supplementary.

12.3.4 Parallel Lines intersected by Transversal Lines

Two lines intersect if they cross each other at a point. For example, at a traffic intersection, two or more streets intersect; the middle of the intersection is the common point between the streets.

Parallel lines are lines that never intersect. For example the tracks of a railway line are parallel. We wouldn't want the tracks to intersect as that would be catastrophic for the train!



All these lines are parallel to each other. Notice the arrow symbol for parallel.

A section of the Australian National Railways Trans-Australian line is perhaps one of the longest pairs of man-made parallel lines.

Longest Railroad Straight (Source: www.guinnessworldrecords.com)
The Australian National Railways Trans-Australian line over the Nullarbor Plain, is 478 km (297 miles) dead straight, from Mile 496, between Nurina and Loongana, Western Australia, to Mile 793, between Ooldea and Watson, South Australia.

A *transversal* of two or more lines is a line that intersects these lines. For example in Figure 12.8, AB and CD are two parallel lines and EF is a transversal. We say $AB \parallel CD$. The properties of the angles formed by these intersecting lines are summarised in the table below.

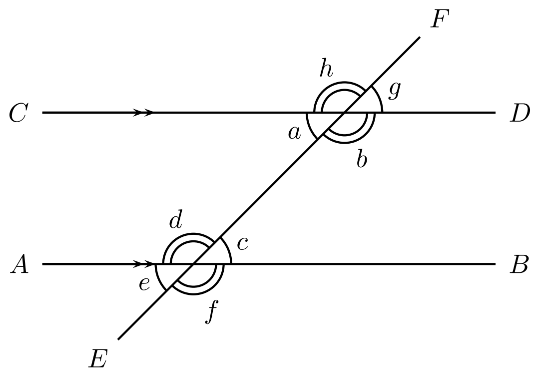
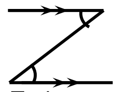
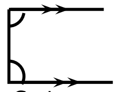
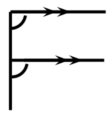


Figure 12.8: Parallel lines intersected by a transversal

Extension: Euclid's Parallel Line Postulate
If a straight line falling on two straight lines makes the two interior angles on the same side less than two right angles (180°), the two straight lines, if produced indefinitely, will meet on that side. This postulate can be used to prove many identities about the angles formed when two parallel lines are cut by a transversal.

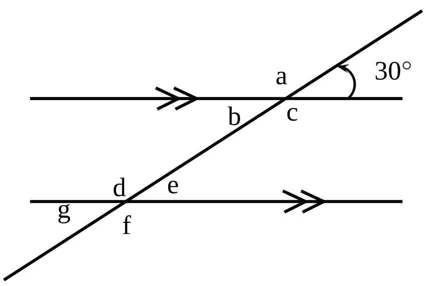
Name of angle	Definition	Examples	Notes
interior angles	the angles that lie inside the parallel lines	a, b, c and d are interior angles	the word *interior* means inside
exterior angles	the angles that lie outside the parallel lines	e, f, g and h are exterior angles	the word *exterior* means outside
alternate interior angles	the interior angles that lie on opposite sides of the transversal	(a,c) and (b,d) are pairs of alternate interior angles, $a = c$, $b = d$	 Z shape
co-interior angles on the same side	co-interior angles that lie on the same side of the transversal	(a,d) and (b,c) are interior angles on the same side. $a + d = 180°$, $b + c = 180°$	 C shape
corresponding angles	the angles on the same side of the transversal and the same side of the parallel lines	(a,e), (b,f), (c,g) and (d,h) are pairs of corresponding angles. $a = e$, $b = f$, $c = g$, $d = h$	 F shape

Important:

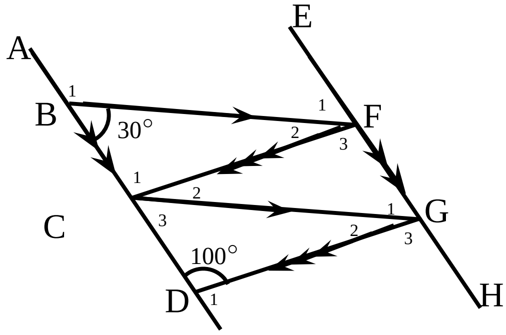
1. If two parallel lines are intersected by a transversal, the sum of the co-interior angles on the same side of the transversal is 180°.
2. If two parallel lines are intersected by a transversal, the alternate interior angles are equal.
3. If two parallel lines are intersected by a transversal, the corresponding angles are equal.
4. If two lines are intersected by a transversal such that any pair of co-interior angles on the same side is supplementary, then the two lines are parallel.
5. If two lines are intersected by a transversal such that a pair of alternate interior angles are equal, then the lines are parallel.
6. If two lines are intersected by a transversal such that a pair of alternate corresponding angles are equal, then the lines are parallel.

Exercise: Angles

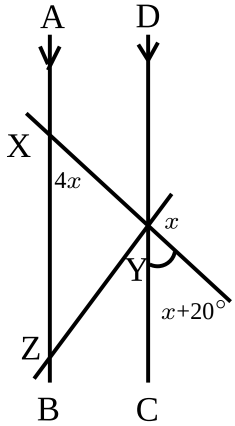
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labeled with letters in the diagram alongside:



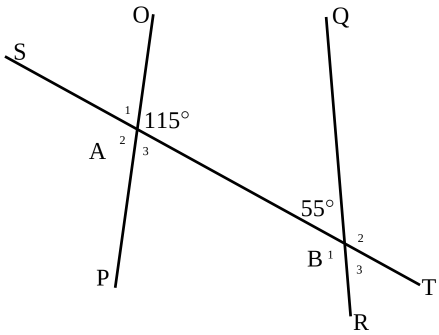
2. Find all the unknown angles in the figure alongside:



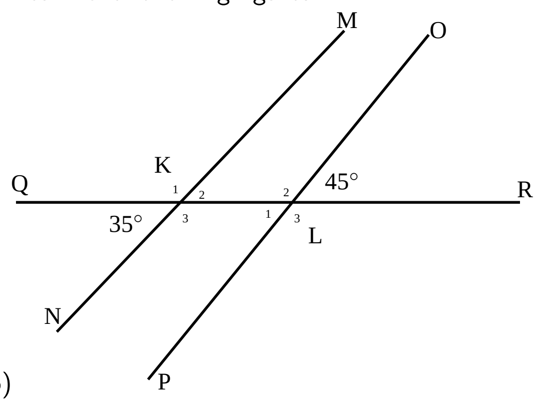
3. Find the value of x in the figure alongside:



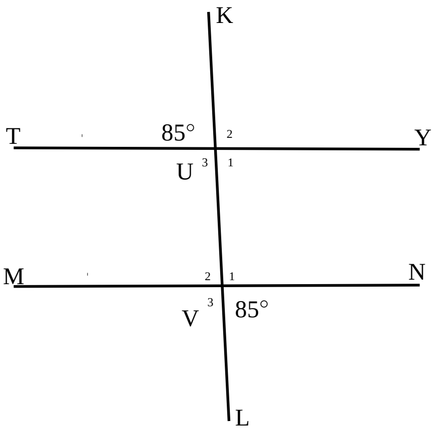
4. Determine whether there are pairs of parallel lines in the following figures.



a)

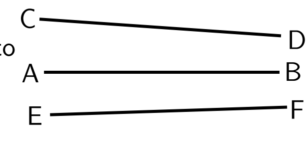


b)



c)

5. If AB is parallel to CD and AB is parallel to EF, prove that CD is parallel to EF:



12.4 Polygons

If you take some lines and join them such that the end point of the first line meets the starting point of the last line, you will get a *polygon*. Each line that makes up the polygon is known as a *side*. A polygon has interior angles. These are the angles that are inside the polygon. The number of sides of a polygon equals the number of interior angles. If a polygon has equal length sides and equal interior angles then the polygon is called a *regular polygon*. Some examples of polygons are shown in Figure 12.9.

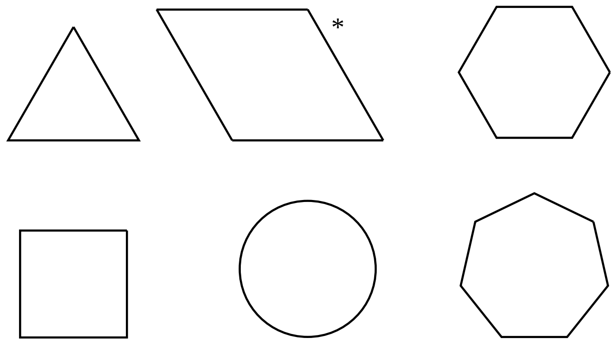


Figure 12.9: Examples of polygons. They are all regular, except for the one marked *

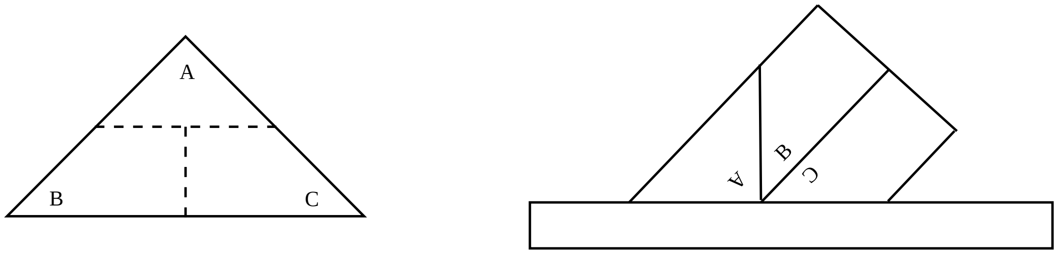
12.4.1 Triangles

A triangle is a three-sided polygon. There are four types of triangles: equilateral, isosceles, right-angled and scalene. The properties of these triangles are summarised in Table 12.2.

Properties of Triangles

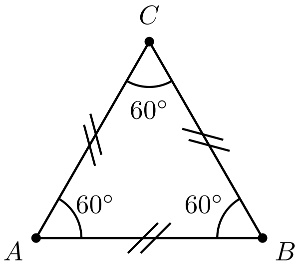
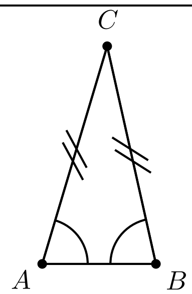
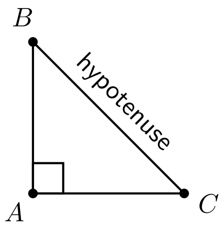
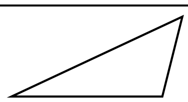
Activity :: Investigation : Sum of the angles in a triangle

1. Draw on a piece of paper a triangle of any size and shape
2. Cut it out and label the angles $\hat{A}$, $\hat{B}$ and $\hat{C}$ on both sides of the paper
3. Draw dotted lines as shown and cut along these lines to get three pieces of paper
4. Place them along your ruler as shown to see that $\hat{A} + \hat{B} + \hat{C} = 180°$



Important: The sum of the angles in a triangle is 180°.

Table 12.2: Types of Triangles

Name	Diagram	Properties
equilateral		All three sides are equal in length and all three angles are equal.
isosceles		Two sides are equal in length. The angles opposite the equal sides are equal.
right-angled		This triangle has one right angle. The side opposite this angle is called the *hypotenuse*.
scalene (non-syllabus)		All sides and angles are different.

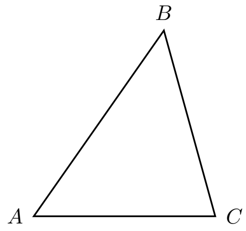


Figure 12.10: In any triangle, $\angle A + \angle B + \angle C = 180°$

Important: Any exterior angle of a triangle is equal to the sum of the two opposite interior angles. An exterior angle is formed by extending any one of the sides.

$B\hat{A}C + B\hat{C}A = C\hat{B}D$ $\quad B\hat{A}C + C\hat{B}A = B\hat{C}D$ $\quad A\hat{B}C + B\hat{A}C = A\hat{C}D$

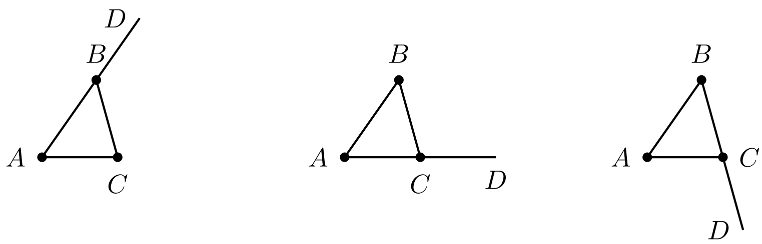
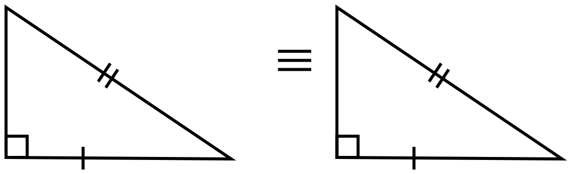
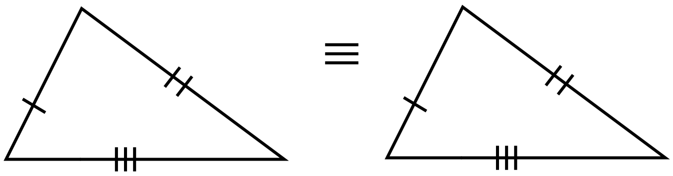
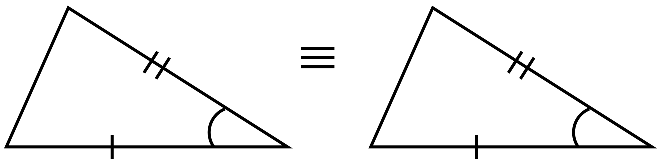
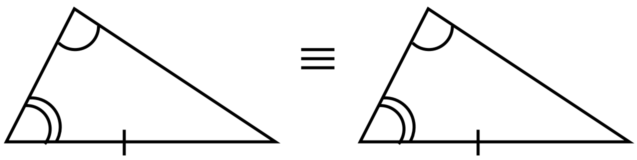
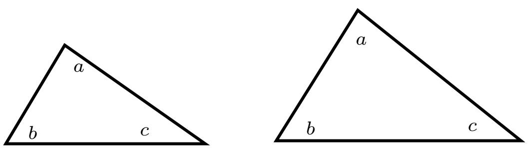
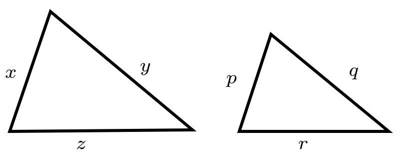


Figure 12.11: In any triangle, any exterior angle is equal to the sum of the two opposite interior angles.

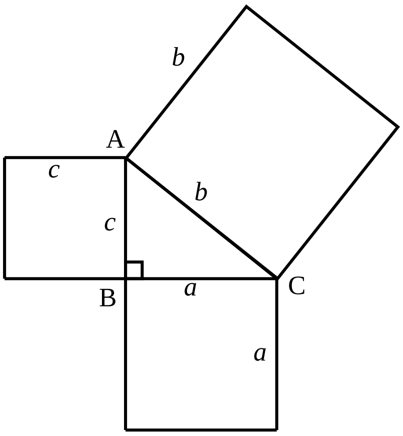
Congruent Triangles

Label	Description	Diagram
RHS	If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the respective side of another triangle then the triangles are congruent.	
SSS	If three sides of a triangle are equal in length to the same sides of another triangle then the two triangles are congruent	
SAS	If two sides and the included angle of one triangle are equal to the same two sides and included angle of another triangle, then the two triangles are congruent.	
AAS	If one side and two angles of one triangle are equal to the same one side and two angles of another triangle, then the two triangles are congruent.	

Similar Triangles

Description	Diagram
If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar.	
If all pairs of corresponding sides of two triangles are in proportion, then the triangles are similar.	 $\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$

The theorem of Pythagoras

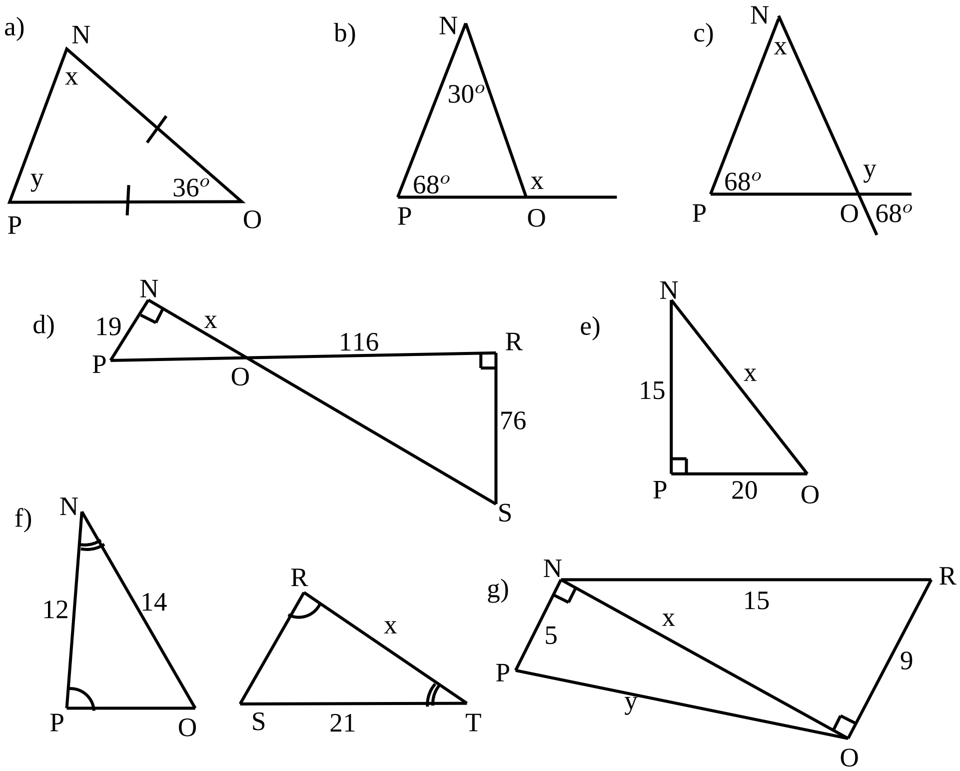


If $\triangle$ABC is right-angled ($\hat{B} = 90°$) then
$b^2 = a^2 + c^2$

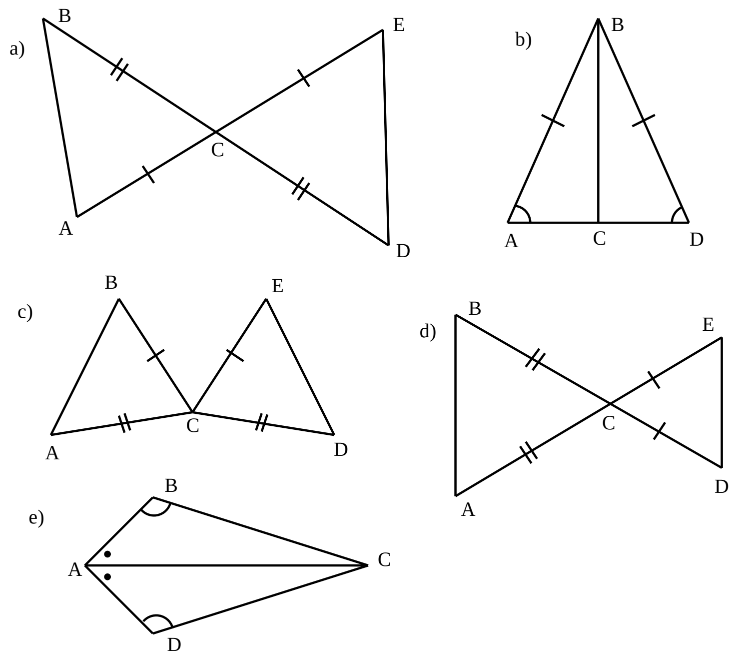
Converse:
If $b^2 = a^2 + c^2$, then
$\triangle$ABC is right-angled ($\hat{B} = 90°$).

Exercise: Triangles

1. Calculate the unknown variables in each of the following figures. All lengths are in mm.



2. State whether or not the following pairs of triangles are congruent or not. Give reasons for your answers. If there is not enough information to make a descision, say why.



12.4.2 Quadrilaterals

A *quadrilateral* is any polygon with four sides. The basic quadrilaterals are the trapezium, parallelogram, rectangle, rhombus, square and kite.

Name of quadrilateral	Figure
trapezium	Figure 12.12
parallelogram	Figure 12.13
rectangle	Figure 12.14
rhombus	Figure 12.15
square	Figure 12.16
kite	Figure 12.17

Table 12.3: Examples of quadrilaterals.

Trapezium

A trapezium is a quadrilateral with one pair of parallel opposite sides. It may also be called a *trapezoid*. A special type of trapezium is the *isosceles trapezium*, where one pair of opposite sides is parallel, the other pair of sides is equal in length and the angles at the ends of each parallel side are equal. An isosceles trapezium has one line of symmetry and its diagonals are equal in length.

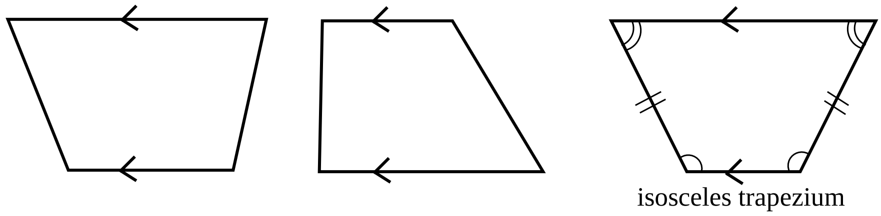


Figure 12.12: Examples of trapeziums.

Parallelogram

A trapezium with both sets of opposite sides parallel is called a *parallelogram*. A summary of the properties of a parallelogram is:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other (i.e. they cut each other in half).

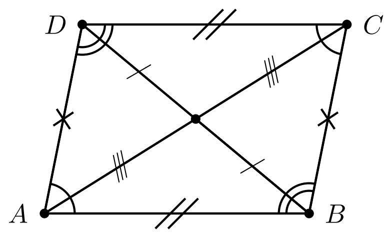


Figure 12.13: An example of a parallelogram.

Rectangle

A *rectangle* is a parallelogram that has all four angles equal to $90°$. A summary of the properties of a rectangle is:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are of equal length equal.
- Both diagonals bisect each other.
- Diagonals are equal in length.
- All angles are right angles.

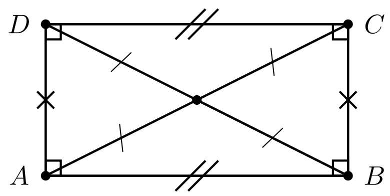


Figure 12.14: Example of a rectangle.

Rhombus

A *rhombus* is a parallelogram that has all four side of equal length. A summary of the properties of a rhombus is:

- Both pairs of opposite sides are parallel.
- All sides are equal in length.

- Both pairs of opposite angles equal.
- Both diagonals bisect each other at $90°$.
- Diagonals of a rhombus bisect both pairs of opposite angles.

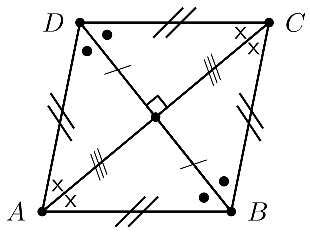


Figure 12.15: An example of a rhombus. A rhombus is a parallelogram with all sides equal.

Square

A *square* is a rhombus that has all four angles equal to 90°.

A summary of the properties of a rhombus is:

- Both pairs of opposite sides are parallel.
- All sides are equal in length.
- All angles are equal to $90°$.
- Both pairs of opposite angles equal.
- Both diagonals bisect each other at $90°$.
- Diagonals are equal in length.
- Diagonals bisect both pairs of opposite angles (ie. all $45°$).

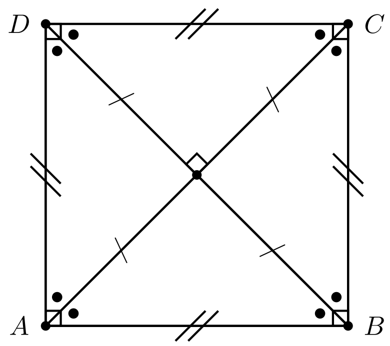


Figure 12.16: An example of a square. A square is a rhombus with all angles equal to 90°.

Kite

A *kite* is a quadrilateral with two pairs of adjacent sides equal.

A summary of the properties of a kite is:

- Two pairs of adjacent sides are equal in length.

- One pair of opposite angles are equal where the angles must be between unequal sides.
- One diagonal bisects the other diagonal and one diagonal bisects one pair of opposite angles.
- Diagonals intersect at right-angles.

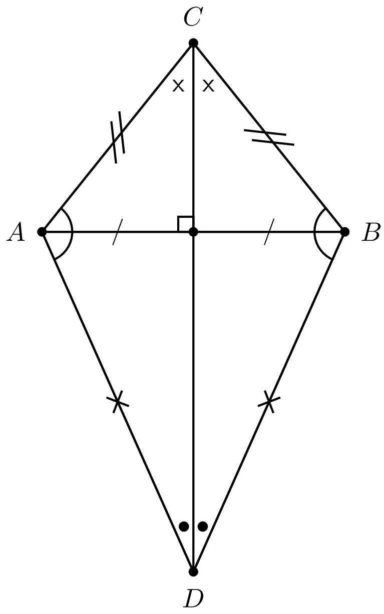


Figure 12.17: An example of a kite.

12.4.3 Other polygons

There are many other polygons, some of which are given in the table below.

Sides	Name
5	pentagon
6	hexagon
7	heptagon
8	octagon
10	decagon
15	pentadecagon

Table 12.4: Table of some polygons and their number of sides.

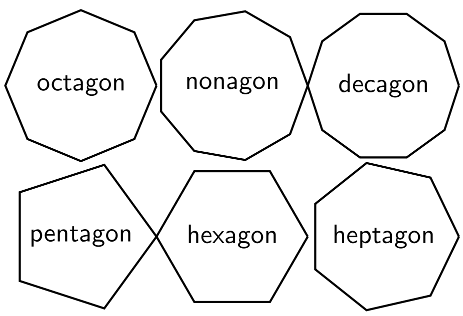


Figure 12.18: Examples of other polygons.

12.4.4 Extra

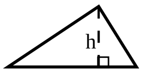
Angles of regular polygons

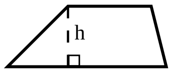
You can calculate the size of the interior angle of a regular polygon by using:

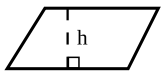
$$\hat{A} = \frac{n-2}{n} \times 180^\circ \qquad (12.1)$$

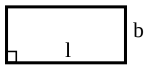
where n is the number of sides and $\hat{A}$ is any angle.

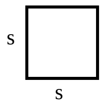
Areas of Polygons

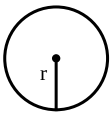
1. Area of triangle: $\frac{1}{2}\times$ base $\times$ perpendicular height 

2. Area of trapezium: $\frac{1}{2}\times$ (sum of $\parallel$ sides) $\times$ perpendicular height 

3. Area of parallelogram and rhombus: base $\times$ perpendicular height 

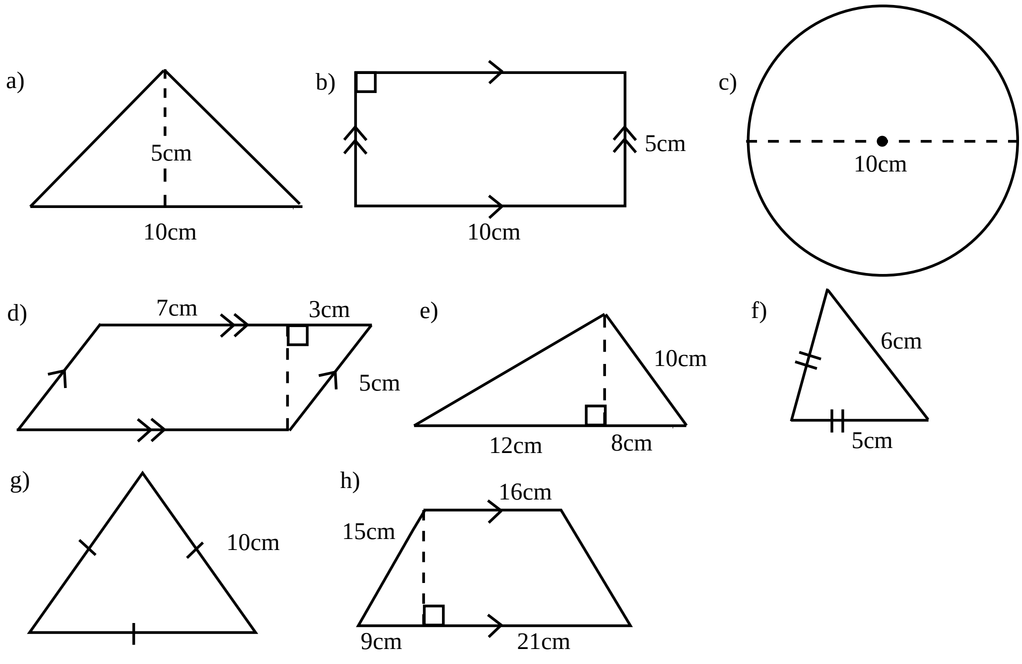
4. Area of rectangle: length $\times$ breadth 

5. Area of square: length of side $\times$ length of side 

6. Area of circle: π x radius^2 

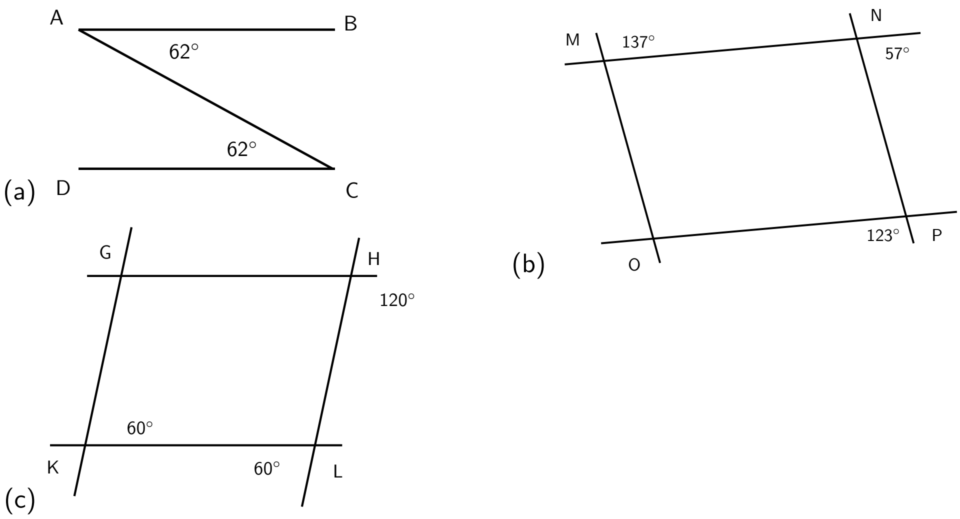
Exercise: Polygons

1. For each case below, say whether the statement is true or false. For false statements, give a counter-example to prove it:
   (a) All squares are rectangles
   (b) All rectangles are squares
   (c) All pentagons are similar
   (d) All equilateral triangles are similar
   (e) All pentagons are congruent
   (f) All equilateral triangles are congruent
2. Find the areas of each of the given figures - remember area is measured in square units (cm^2, m^2, mm^2).

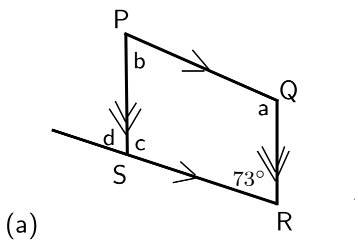


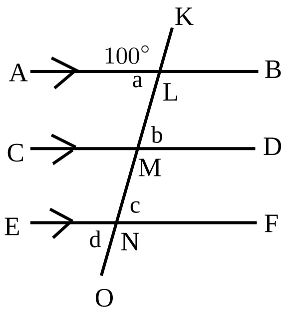
12.5 Exercises

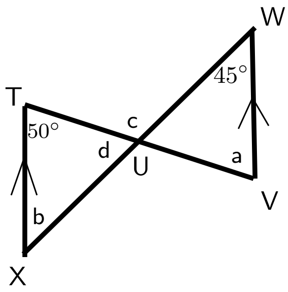
1. Find all the pairs of parallel lines in the following figures, giving reasons in each case.



2. Find a, b, c and d in each case, giving reasons.



(b)


(c)


(a) Which of the following claims are true? Give a counter-example for those that are incorrect.

i. All equilateral triangles are similar.

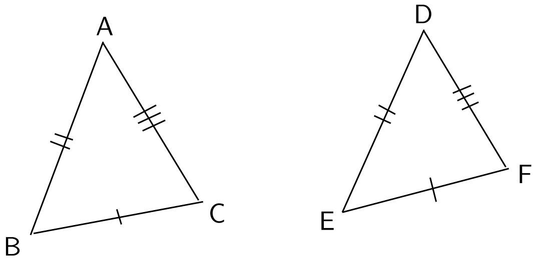
ii. All regular quadrilaterals are similar.

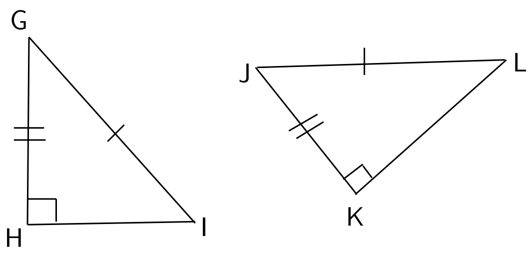
iii. In any $\triangle ABC$ with $\angle ABC = 90°$ we have $AB^3 + BC^3 = CA^3$.

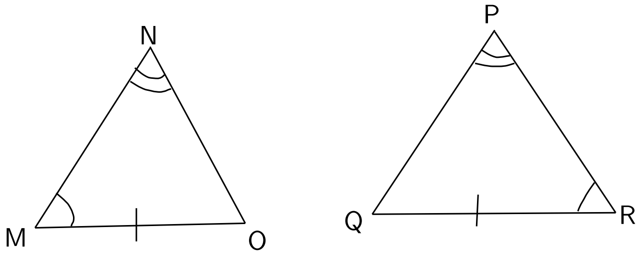
iv. All right-angled isosceles triangles with perimeter 10 cm are congruent.

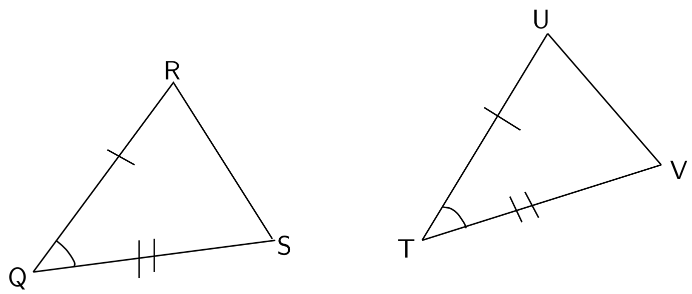
v. All rectangles with the same area are similar.

(b) Say which of the following pairs of triangles are congruent with reasons.

i.


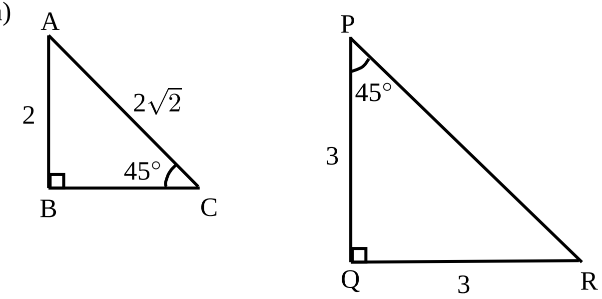
ii.


iii.


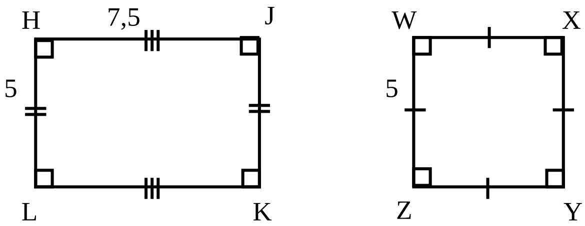
iv.


(c) For each pair of figures state whether they are similar or not. Give reasons.

(a)

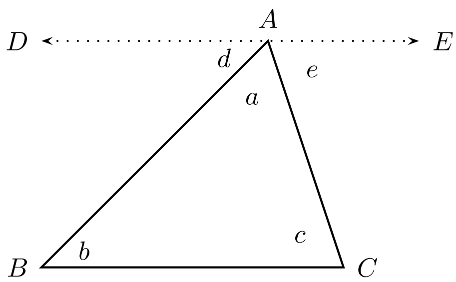


(b)



12.5.1 Challenge Problem

1. Using the figure below, show that the sum of the three angles in a triangle is 180°. Line DE is parallel to BC.



Appendix A

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